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THE FORMATION AND STABILITY OF NORMAL SHOCK WAVES IN CHANNEL FLOWS

By Arthur Kantrowitz

Langley Aeronautical Laboratory
Langley Field, Va.

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SHOCK WAVES IN CHANNEL FLOWS

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SUMMARY

There is experimental evidence that channel flows involving shock-free deceleration through the speed of sound are unstable. An analysis of nonviscous unsteady channel flow has been made to gain some insight into this apparent instability, to provide some information on the factors determining the minimum shock intensity for a stable flow, and to study in general the formation and motion of shocks in channel flows. It is hoped that this treatment will be useful in two ways. First, the theory will have direct application to the study of normal shocks occuring in supersonic nozzles and diffusers. Second, it might be useful in leading to the solution of the more difficult problems of the formation and stability of shocks in two and three dimensions.

The analysis is divided into two parts. In part I, Riemann's theory of the propagation of finite-amplitude disturbances in a homogeneous medium is extended to the case where upstream-moving pulses are superposed on a decelerating transonic channel flow. Shock waves are formed as the pulse is propagated, as in Riemann's problem. If the intensity of the shocks is assumed to be very small, the pulse approaches a "trapped" state in which a portion of the decelerating channel flow is converted to an accelerating flow which is an alternate steady-state solution for the channel. To the order of accuracy of the theory of part I, the trapped pulse becomes stationary in the sonic region of the channel and, thus, the flow has neutral stability.

In part II, the assumption of weak shocks is dropped and the motion of shock waves is considered more accurately. It is found that trapped expansion pulses are consumed by the shock motion and trapped compression pulses inevitably grow. Thus, it is concluded that smooth transonic deceleration is unstable to compression pulses coming from the rear of the channel.

A similar analysis is made for the case where the shock wave forms a part of the equilibrium channel flow. The shock position is found to be stable in diverging channels and unstable in converging channels. This analysis leads also to suggestions for the design and testing of supersonic diffusers.

INTRODUCTION

Experimental investigation of transonic channel flows has established a striking difference between the processes of acceleration and deceleration through the speed of sound. In a de Laval nozzle, a gas is accelerated to the velocity of sound in the converging (that is, converging downstream) part of the nozzle and is accelerated further to supersonic velocities in the diverging part. If the nozzle is shaped to avoid the condensation of compression waves, the acceleration through the speed of sound can be made very smoothly (without shock waves).

As far as is known, the reverse flow, smooth deceleration through the speed of sound in a channel flow, has never been observed. The experimental situation can be summed up briefly as follows: In the process of starting the supersonic flow, a normal shock always forms ahead of the converging part of the channel (supersonic diffuser). By suitably changing the velocities and pressures at the ends of the diffuser, the normal shock can be made to jump to the diverging part. With further changes in velocities, pressures, and/or geometry, the normal shock can be pushed up the diverging channel and its intensity reduced somewhat. However, before the shock can be made to disappear, it suddenly (to visual observation) jumps to a position ahead of the converging part of the diffuser and the starting process must begin again. The experiments strongly indicate, therefore, that smooth deceleration through the speed of sound in a channel flow is unstable.

An analysis of this apparent instability is the central problem of this paper. Specifically, the stability of transonic decelerating channel flow to a special type of disturbance coming from the rear of the channel is considered. The analysis is restricted to channels of slowly varying cross section so that a quasi one-dimensional treatment can be used. This treatment is based on Riemann's treatment (reference 1) of plane waves superposed on a homogeneous medium. A part of Riemann's paper will be paraphrased herein to put his results into a form suitable for extension to channel-flow problems.

The author is very grateful for the helpful discussion of these problems with Mr. A. Kahane and Mr. Lester Lees.

RIEMANN'S TREATMENT OF THE PROPAGATION OF FINITE PLANE DISTURBANCES IN A HOMOGENEOUS MEDIUM

In 1859, Riemann published his great paper (reference 1) on the propagation of plane aerial waves of finite amplitude. Starting with the one-dimensional equations of motion and continuity and assuming isentropic flow, he shows (among other things) that: (1) a disturbance initially confined to a finite region will spread out into waves propagated in both directions from the source, and (2) that compression fronts in the propagated waves will steepen indefinitely, and will eventually form a compression shock.

Riemann starts with the one-dimensional equations of motion and continuity (assuming isentropic flow of a perfect gas):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a^2\frac{\partial \log \rho}{\partial x} = 0 \tag{1}$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{2}$$

(Symbols are defined in appendix A.)

He then multiplies equation (2) by $\pm\frac{a}{\rho}$ and adds to equation (1) obtaining

$$\frac{\partial u}{\partial t} \pm a\frac{\partial \log \rho}{\partial t} + (u \pm a)\left(\frac{\partial u}{\partial x} \pm a\frac{\partial \log \rho}{\partial x}\right) = 0 \tag{3}$$

Assuming all changes to be isentropic and introducing $\gamma = 1.4$ (the value for room-temperature air) yields

$$d \log \rho = \frac{2\, da}{(\gamma - 1)a} = 5\frac{da}{a} \tag{4}$$

Hence, $a\, d \log \rho = d\, 5a$. Using this result permits equation (3) to be written

$$\frac{\partial}{\partial t}(u \pm 5a) + (u \pm a)\frac{\partial}{\partial x}(u \pm 5a) = 0 \tag{5}$$

It is convenient to introduce the symbols $P = u + 5a$ and $Q = u - 5a$ which occur frequently in these equations. With these symbols $u + a = \frac{3P + 2Q}{5}$ and $u - a = \frac{2P + 3Q}{5}$.

The equations (5) can be written

$$\frac{\partial P}{\partial t} + \frac{3P + 2Q}{5} \frac{\partial P}{\partial x} = 0 \tag{6}$$

and

$$\frac{\partial Q}{\partial t} + \frac{2P + 3Q}{5} \frac{\partial Q}{\partial x} = 0 \tag{7}$$

If the disturbances are not too large $(|u| < a)$ these equations represent disturbances moving in the $\pm x$ directions, respectively, since for this case $\frac{3P + 2Q}{5} > 0$ and $\frac{2P + 3Q}{5} < 0$. If, for example, there are no disturbances in a given region moving in the $-x$ direction, then $Q = \text{Const.} = -5a_s$ where a_s is the velocity of sound in air at rest in this region. In this case, it can be seen from equation (7) that Q remains constant, since $\frac{\partial Q}{\partial x} = 0$, even in the presence of a P wave. Thus, waves moving in one direction in a homogeneous medium do not reflect waves moving in the opposite direction.

In the analysis of channel flow, finite pulses coming from the rear of the channel will be assumed. The term "finite pulse" is used to denote a finite disturbed region bounded on both sides by undisturbed fluid (or equilibrium flow in a channel-flow problem). A finite pulse will consist of at least one phase where the pressure drops, which will be called the expansion phase, and at least one compression phase where the pressure rises with time.

It can be shown that the integral $\int P\,dx$ (or $\int Q\,dx$), which will be called the pulse area, taken over the whole of a finite pulse does not change with time when pulses moving in only one direction are present. Integrating equation (6) with respect to x gives

$$\frac{\partial}{\partial t} \int P\,dx + \int \frac{3P + 2Q}{5}\,dP = 0$$

Since P vanishes at both ends of the pulse, the second integral is 0 for $Q = \text{Const.}$ and $\int P\,dx$ (or similarly $\int Q\,dx$) is constant in time[1].

Riemann used equations (6) and (7) to show that compression fronts in these waves steepen to form shocks while expansion waves flatten out indefinitely. This information can be obtained for the case where only waves moving in one direction are present by a different approach which will be readily extended to the problems considered later by examining equation (7). Differentiating equation (7) with respect to x, assuming $P = \text{Const.}$, and introducing $\epsilon = \frac{\partial Q}{\partial x}$, there is obtained

$$\frac{\partial \epsilon}{\partial t} + \frac{2P + 3Q}{5}\frac{\partial \epsilon}{\partial x} = -\frac{3}{5}\epsilon^2 \tag{8}$$

For the case where only Q disturbances are present the quantity ϵ may be understood physically as twice the velocity gradient in the disturbance since $u = \frac{P + Q}{2}$.

Consider now that the point under observation moves with the propagation velocity and let $\frac{\delta \epsilon}{dt}$ denote the rate of change of ϵ for this point. Now

$$\frac{\delta \epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \frac{\delta x}{dt}\frac{\partial \epsilon}{\partial x}$$

where $\frac{\delta x}{dt}$ is the local propagation velocity $\frac{2P + 3Q}{5}$. Hence,

[1]A more general form of this theorem can be obtained as follows: let $f(P)$ be an arbitrary well-behaved function of P. Multiply equation (6) by $\frac{df(P)}{dP}$ and integrate as before. It follows that $\frac{\partial}{\partial t}\int f(P)dx = 0$ where the integration is over the whole of a finite pulse. Thus, the mass, impulse or energy carried by a pulse will also remain constant and any of these quantities could be used. The quantity $\int Pdx$ is convenient since P appears directly in the equations.

equation (8) can be written

$$\frac{\delta\epsilon}{dt} = -\frac{3}{5}\epsilon^2$$

or

$$\frac{\epsilon}{\bar{\epsilon}} = \frac{1}{\frac{3}{5}t\bar{\epsilon} + 1}$$

where $\bar{\epsilon}$ is the value of ϵ at $t = 0$. Now, if $\bar{\epsilon}$ is positive, ϵ decreases toward zero as the wave is propagated. If $\bar{\epsilon}$ is negative, $|\epsilon|$ increases indefinitely, becoming infinite at $t = -\frac{5}{3\bar{\epsilon}}$.

For the case where only Q waves are present in the region under consideration, positive and negative values of ϵ can be identified from the fundamental equations as expansion and compression waves, respectively. Thus, compression waves steepen to form shocks and rarefaction waves flatten out indefinitely.

GENERAL PLAN OF THE CHANNEL-FLOW ANALYSIS

The general plan of this treatment can best be explained by considering the central problem, the stability of smooth deceleration through the speed of sound, from a physical point of view. In such a flow, disturbances starting in the rear of the channel will be trapped in the sonic region. The presence of this disturbance trap constitutes a distinct difference between this flow and smooth flows known to be stable.

Two types of processes which govern the behavior of pulses in the sonic region are considered separately in parts I and II of this paper. First, there are isentropic processes similar to the propagation of waves of finite amplitude considered by Riemann. Shocks form in the channel flow as in Riemann's case. If these shock waves are assumed to be of very small intensity, it is shown in part I that a pulse approaches a stationary state. Thus, the amplitude, shape, and position of a pulse approach assymptotically what will be called a "trapped pulse." The motion of the shock waves in these trapped pulses is considered more accurately in part II, and it is found that the shock moves and causes the pulse to be consumed or to grow, depending on its sign. In cases where the intensity of the shocks involved is not too large, the consumption process is much slower than the trapping process. In these cases, it is possible to obtain a fairly complete history of the behavior of a pulse without considering the interaction of these two processes. Thus, in part I, the rapid trapping process is considered and the slow consumption of the pulse by shocks is neglected. In part II, the consumption of fully trapped pulses by shock waves is discussed.

A similar analysis is made for the case where a shock wave forms part of the initial channel flow. In this case, disturbances from the rear serve to move the shock to a new position and, if changes in shock strength are neglected, it is found that the pulses in this case are also trapped as shock displacements (shown in part I). Here again, the slower effects on this shock displacement are discussed in part II, the changes in shock strength being included. In this way, the stability of shock waves in channel flows is studied.

The boundary-layer effects are neglected entirely in this paper. It is well known, however, that such effects are important in channel flows involving shock waves. The results obtained can therefore, at best, be only qualitatively similar to experimental results. Perhaps it will be possible eventually to add boundary-layer effects for cases where they are important, thus following a procedure frequently used in gas dynamics.

It should be pointed out that information of the type sought in this analysis cannot be found by introducing the acoustic approximation; that is, that the disturbance velocities are negligible compared with the propagation velocity. In the sonic region of a channel, the propagation velocity of an upstream-moving pulse is close to zero. On the other hand, if the acoustic approximation is introduced, the disturbance velocities tend to become infinite in the sonic region (reference 2). Thus, it is clear that the acoustic approximation cannot be used to study the development of upstream-moving pulses in the sonic region.

I – THE TRAPPING OF PULSES AND THE FORMATION OF SHOCKS IN A DECELERATING TRANSONIC CHANNEL FLOW

The Propagation of Finite Plane Disturbances Superposed on a Channel Flow

General equations.– An analysis similar to Riemann's can be applied to the case where plane disturbances are superposed on a one-dimensional flow field. (A one-dimensional flow field is one where the variations in velocity, density, etc., perpendicular to the flow direction, are negligible compared with those in the flow direction.) Consider the field of flow in a straight channel of slowly varying cross section S. The equation of continuity becomes

$$S\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u S)}{\partial x} = 0$$

or

$$\frac{\partial \log \rho}{\partial t} + u\frac{\partial \log \rho}{\partial x} + \frac{\partial u}{\partial x} + u\frac{d \log S}{dx} = 0 \qquad (9)$$

Multiplying equation (9) by $\pm a$ and adding to equation (1) yields

$$\frac{\partial}{\partial t}(u \pm 5a) + (u \pm a)\frac{\partial}{\partial x}(u \pm 5a) \pm ua\frac{d \log S}{dx} = 0 \qquad (10)$$

An equilibrium (steady) flow in the channel with quantities denoted by subscript o will be assumed. Primed quantities will denote departures from this equilibrium. Thus,

$$\left.\begin{aligned} u &= u_o + u' \\ a &= a_o + a' \end{aligned}\right\} \qquad (11)$$

By substitution of equations (11) in equation (10), there is obtained

$$\frac{\partial}{\partial t}(u' \pm 5a') + (u_o \pm a_o)\frac{d}{dx}(u_o \pm 5a_o) \pm u_o a_o\frac{d \log S}{dx}$$

$$+ \left[(u_o \pm a_o) + (u' \pm a')\right] \frac{\partial(u' \pm 5a')}{\partial x} + (u' \pm a')\frac{d(u_o \pm 5a_o)}{dx}$$

$$\pm (u'a' + u'a_o + a'u)\frac{d \log S}{dx} = 0 \qquad (12)$$

Note that, where $M_o = \dfrac{u_o}{a_o}$,

$$\frac{d \log S}{dx} = \frac{1}{u_o}(M_o^2 - 1)\frac{du_o}{dx}$$

and

$$\frac{d(5a_o)}{dx} = -M_o\frac{du_o}{dx}$$

Dropping the second and third terms of equation (12), which cancel each other (equation (10) for steady-flow case), gives

$$\frac{\partial}{\partial t}(u' \pm 5a') + \left[(u_o \pm a_o) + (u' \pm a')\right]\frac{\partial(u' \pm 5a')}{\partial x}$$

$$+ (u' \pm a')(1 \mp M_o)\frac{du_o}{dx} \pm (M_o^2 - 1)\left(\frac{u'a'}{u_o} + \frac{u'}{M_o} + a'\right)\frac{du_o}{dx} = 0 \qquad (13)$$

If the quantities P and Q are introduced, where now $P = u' + 5a'$ and $Q = u' - 5a'$, the equations (13) become

$$\frac{\partial P}{\partial t} + \left(u_o + a_o + \frac{3P + 2Q}{5}\right)\frac{\partial P}{\partial x} + (1 - M_o)\left(\frac{3P + 2Q}{5}\right)\frac{du_o}{dx}$$

$$+ (M_o^2 - 1)\left(\frac{P^2 - Q^2}{20u_o} + \frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\frac{du_o}{dx} = 0 \qquad (14)$$

and

$$\frac{\partial Q}{\partial t} + \left(u_o - a_o + \frac{2P + 3Q}{5}\right)\frac{\partial Q}{\partial x} + (1 + M_o)\left(\frac{2P + 3Q}{5}\right)\frac{du_o}{dx}$$

$$- (M_o^2 - 1)\left(\frac{P^2 - Q^2}{20u_o} + \frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\frac{du_o}{dx} = 0 \qquad (15)$$

Reflections in a velocity field.- The equations (14) and (15) indicate that disturbances will be propagated upstream and downstream in the channel flow as was the case in Riemann's problem. Because of the last two terms in these equations, however, a pulse moving in one direction will reflect disturbances moving in the other direction. Perhaps it would be possible to solve these equations numerically for any particular case by the method of characteristics. It will, however, serve the general purposes of this paper better to derive general approximations rather than more accurate numerical results for specific cases.

In part I of this treatment, the reflections will be neglected; but before this assumption is introduced, an evaluation of the order of magnitude of these reflections will be made to provide an indication of the error thus introduced.

First, consider a pulse moving upstream into an equilibrium velocity field; that is, containing no downstream-moving disturbances. Consider a point moving with the downstream velocity of propagation $u + a = u_o + a_o + \frac{3P + 2Q}{5}$. The rate of change of P at this observation point will be denoted by $\frac{\delta P}{dt}$ and is equal to $\frac{\partial P}{\partial t} + \frac{\delta x}{dt}\frac{\partial P}{\partial x}$. The rate of growth of P while this point crosses the Q pulse is thus given by the sum of the first two terms of equation (14); that is

$$\frac{\partial P}{dt} = \frac{\partial P}{\partial t} + \left(u_o + a_o + \frac{3P + 2Q}{5}\right)\frac{\partial P}{\partial x}$$

$$= -\frac{du_o}{dx}\left[(1 - M_o)\frac{3P + 2Q}{5} + (M_o^2 - 1)\left(\frac{P^2 - Q^2}{20u_o} + \frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\right]$$

In order to obtain a simple estimate of the reflections produced in the sonic region of a channel, it will be convenient to introduce the following approximations in the right-hand side of this equation; (1) P is negligible compared with Q, (2) $M_o + 1 = 2M_o = 2$, (3) $\frac{Q^2}{20u_o}$ is negligible compared with $\frac{Q}{2}$. Thus, there is obtained

$$\frac{\delta P}{dt} = \frac{du_o}{dx}\,Q\,\frac{2}{5}(1 - M_o)$$

Now, integrating over the whole of an upstream moving pulse gives for the value of P at the rear (which will be the maximum value of P if Q is all of one sign throughout the pulse)

$$P_{rear} = \frac{2}{5}\int_{Q_{pulse}} (1 - M_o)Q\frac{du_o}{dx}\, dt$$

Since the propagation velocity is close to $2a_o$, $dt = \frac{dx}{2a_o}$ (for the point under observation), and approximately

$$P_{rear} = \frac{2}{5}\int_{Q_{pulse}} \frac{(1 - M_o)Q}{2a_o}\frac{du_o}{dx}\, dx$$

For example, for a square pulse - that is, a pulse for which Q = Const. from a point where the steady velocity is u_o to a point where it is $u_o + \Delta u_o$, there is obtained

$$P_{rear} = Q\,\frac{\Delta u_o}{a_o}\,\frac{1}{5}(1 - M_o)_{av}$$

where $(1 - M_o)_{av}$ means an average over the region of the pulse. The relative magnitudes of P and Q in this problem can be illustrated by a numerical example. If $(1 - M_o)_{av} = 0.1$ and $\frac{\Delta u_o}{a_o} = 0.1$, then $P_{rear} = - \frac{Q}{500}$.

It thus appears that for short upstream-moving pulses in the sonic region, it will be possible to neglect P in comparison with Q in equation (15) if there are no downstream-moving disturbances present initially.

Conservation of pulse area.- If reflections are neglected, the propagation of an upstream-moving or Q pulse can be studied with the aid of equation (15) alone. Since this treatment is primarily concerned with the sonic region of channels, it will be possible to obtain simple approximations by neglecting the fourth term of equation (15), which contains the factor $(M_o^2 - 1)$, in comparison with the third term. Eliminating this term, replacing $1 + M_o$ by 2 in the third term, and neglecting P, compared with Q, gives for equation (15)

$$\frac{\partial Q}{\partial t} + \left(u_o - a_o + \frac{3Q}{5}\right)\frac{\partial Q}{\partial x} + \frac{6}{5} Q\frac{du_o}{dx} = 0 \qquad (16)$$

From Riemann's equations, it was shown that the quantity $\int Q\,dx$ (pulse area) is constant in the isentropic propagation of pulses in a homogeneous gas. By the use of the approximate equation (16), it can easily be shown that the same rule is approximately true for propagation in a flow field. Equation (16) can also be written

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left[Q\left(\frac{3}{5}\frac{Q}{2} + u_o - a_o\right)\right] = 0 \qquad (17)$$

as can readily be verified by differentiation (for values of a close to 1).

Integrating this equation with respect to x over the range x_a to x_b gives

$$\frac{\partial}{\partial t}\int_{x_a}^{x_b} Q\,dx + \left[Q\left(\frac{3}{5}\frac{Q}{2} + u_o - a_o\right)\right]_{x_a}^{x_b} = 0 \qquad (18)$$

If the integration is taken over the whole of a finite pulse, the second term vanishes and the pulse area $\int Qdx$ remains constant to the order of accuracy of equation (16). It can also be seen by taking the point x_a ahead of the pulse where $Q = 0$ in equation (18) that, at an arbitrary point x_b, Q pulse area flows through at the rate $\left[-Q\left(\frac{3}{5}\frac{Q}{2} + u_o - a_o\right)\right]_{x_b}$.

It can be shown that, as long as only upstream-moving disturbances are present, this rule - conservation of pulse area in upstream-moving pulses - is not affected by the presence of shocks (if downstream-moving disturbances produced by the shocks are neglected). For the case where the shock is not part of the initial equilibrium flow (that is where it develops in an upstream-moving disturbance) and $P = 0$, it is shown in appendix B (equation (B4)) that the velocity U of propagation of a shock wave is

$$U = u_o - a_o + \frac{3}{10}\left(Q_1 + Q_2\right) \qquad (19)$$

where values of Q before and after the shock are denoted by Q_1 and Q_2, respectively.

If now it can be shown that the pulse area entering the shock per unit time is equal to that leaving the shock, the pulse conservation rule will be satisfied. The pulse area entering or leaving the shock is the sum of the pulse area passing through a stationary point and the pulse area overtaken by the shock; thus, it must be shown that

$$UQ_1 - Q_1\left(\frac{3}{5}\frac{Q_1}{2} + u_o - a_o\right) = UQ_2 - Q_2\left(\frac{3}{5}\frac{Q_2}{2} + u_o - a_o\right) \quad (20)$$

If the expression (19) is substituted for U, this equation is identically satisfied, and thus, this extension of the rule is established.

Another similar case which will be of interest in connection with later considerations is that for an upstream-moving pulse of small amplitude interacting with a shock that is part of the initial equilibrium flow. This case is illustrated in figure 1. It is necessary to assume that the shock intensity is low (equilibrium-shock Mach number close to 1) to get a simple result. It is also necessary to neglect the variations with x of the local equilibrium velocities and velocities of sound, in comparison with the disturbance velocities. These changes, which are due to the changing channel area, are responsible for the stability of position of the shock wave. Thus, in other words, the effects due to the stability of the shock wave are neglected as compared with the disturbance effects. The circumstances under which this assumption is permissible will become clearer from the results of part II. If the stability of a normal shock is neglected, the velocity with which it moves under the influence of a small upstream-moving disturbance is calculated in appendix B (equation (B9)) to be simply

$$U = \frac{3}{10}\left(Q_1 + Q_2\right) \quad (21)$$

In addition to the contributions to the pulse area which were present in the previous problem, pulse area is contributed by the fact that the shock is merely out of position. The equation which must now be satisfied to demonstrate conservation of pulse area is

$$UQ_1 - Q_1\left(\frac{3}{5}\frac{Q_1}{2} + u_{1_o} + a_{1_o}\right) + 2U\left(u_{1_o} - u_{2_o}\right)$$

$$= UQ_2 - Q_2\left(\frac{3}{5}\frac{Q_2}{2} + u_{2_o} - a_{2_o}\right) \tag{22}$$

where the subscripts 1_o and 2_o denote equilibrium conditions before and after the shock, respectively. Noting that approximately (see appendix B)

$$\frac{u_{2_o} - a_{2_o}}{u_{1_o} - u_{2_o}} = -\frac{3}{5}$$

and

$$\frac{u_{1_o} - a_{1_o}}{u_{1_o} - u_{2_o}} = +\frac{3}{5}$$

for Mach numbers close to 1 and introducing the shock velocity (equation (21)), satisfies equation (22) identically. Thus, the rule of conservation of pulse area appears to be a useful approximation with considerable generality.

<u>The pulse shape</u>.- The shape of a pulse (that is, Q as a function of x) advancing upstream in a velocity field, in the absence of downstream-moving disturbances, can be studied in a flow field by considering its x derivative $\frac{\partial Q}{\partial x} = \epsilon$. The variation of this quantity with time can be found by differentiating equation (16) with respect to x.

$$\frac{\partial \epsilon}{\partial t} + \left(u_o - a_o + \frac{3}{5}Q\right)\frac{\partial \epsilon}{\partial x} + \frac{3}{5}\epsilon\left(\epsilon + 4\frac{du_o}{dx}\right) + \frac{6}{5}Q\frac{d^2u_o}{dx^2} = 0$$

For the case where the equilibrium velocity gradient is constant, that is $\frac{d^2u_o}{dx^2} = 0$, the behavior of the pulse shape can be studied simply from this differential equation. Consider the point under observation to move with the wave, that is, move with the local propagation velocity. Let $\frac{\delta\epsilon}{dt}$ be the rate of change of ϵ when the point under observation moves with the propagation velocity; then the preceding equation becomes

$$\frac{\delta\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + \left(u_o - a_o + \frac{3}{5}Q\right)\frac{\partial\epsilon}{\partial x} = -\frac{3}{5}\epsilon\left(\epsilon + 4\frac{du_o}{dx}\right) \qquad (23)$$

The variables in equation (23) can be separated. If the dimensionless parameters $E = \frac{\epsilon}{4\frac{du_o}{dx}}$ and $T = \frac{du_o}{dx}t$ are introduced, equation (23) can be integrated for $\frac{du_o}{dx}$ constant to give

$$E = \frac{1}{\left(1 + \frac{1}{\bar{E}}\right)e^{\frac{12}{5}T} - 1} \qquad (24)$$

where $\bar{E}$ is the dimensionless slope of the wave at the time $t = 0$. Positive values of T in equation (24) will represent accelerating equilibrium-velocity fields $\left(\frac{du_o}{dx} > 0\right)$ and negative values of T will represent decelerating fields. Also, in accelerating fields, positive values of E will represent expansions and negative values will represent compressions, and the opposite is true for decelerating fields. Equation (24) is plotted in figure 2 for various values of $\bar{E}$.

It can be seen from this figure that the results are considerably different from Riemann's. Thus, in an accelerating field, compression waves must have a certain initial steepness, $\bar{E} < -1$, or they will flatten out in time. On the other hand, in a decelerating field, expansion waves, instead of flattening out (as occurs in Riemann's case), approach a slope given by $E = -1$. It should be noted that $E = -1$ corresponds to a total velocity gradient $\frac{du}{dx} = -\frac{du_o}{dx}$.

<u>"Trapped" pulses</u>.- The approximate relations which have been developed in the preceding sections can be used to show that a pulse moving upstream in a uniformly decelerating flow approaches

a stationary state. Thus, the pulse area remains constant, the expansion phases approach the slope given by $E = -1$, and the compression phases become shocks. As these developments take place, the pulse advances to a position where the propagation velocities of the shocks approach zero. When this position is reached the pulse will be said to be "trapped." It should be pointed out that the trapped pulse is an idealization and would not exist in an exact treatment. The introduction of this concept makes it possible to divide the history of a pulse into two periods which can be considered separately. The trapping of compression and expansion pulses in a decelerating transonic channel flow is shown schematically in figure 3.

The asymptotic result for expansion phases of trapped pulses $E = -1$ is compared in figure 4 with the exact accelerating equilibrium channel flow. It can be seen from this figure that if all the velocities involved are close to the local speed of sound, the expansion phases are very close to the accelerating steady flow. It seems reasonable to infer from figure 4 that the differences between the two curves are due to the approximations introduced in deriving equation (16) from the exact equations (14) and (15) (neglect of reflections and $M_o^2 - 1$ terms). In other words, it seems reasonable to infer that an exact calculation from equations (14) and (15) would yield the exact accelerating steady flow. This inference will be adopted in part II and the exact accelerating flow will be used in place of the $E = -1$ result in calculations of the shock velocities.

It will be valuable to calculate the order of magnitude of the times in which the first-order trapping processes take place. These times will be compared with times in which the slower processes of part II take place. For example, the leading edge of an expansion wave propagates upstream with the velocity $u_o - a_o$. If x is the distance of this leading edge from the sonic point of a channel, then in a uniformly decelerating field (for M_o close to 1) $u_o - a_o = \frac{6}{5}\frac{du_o}{dx}x$. The equation for the approach of the leading edge to the sonic point is thus,

$$\frac{dx}{dt} = \frac{6}{5}\frac{du_o}{dx}x$$

The relaxation time for the approach is therefore $\dfrac{1}{\frac{6}{5}\left(-\dfrac{du_o}{dx}\right)}$.

A relaxation time for the slope of an expansion wave to approach its equilibrium slope can similarly be found from equation (23). If the slope is close to the equilibrium value $\epsilon = -4 \frac{du_o}{dx}$, equation (23) can be written

$$\frac{d\left(\epsilon + 4 \frac{du_o}{dx}\right)}{dt} = \frac{12}{5} \frac{du_o}{dx}\left(\epsilon + 4 \frac{du_o}{dx}\right)$$

The relaxation time for this process is then $\frac{1}{\frac{12}{5}\left(-\frac{du_o}{dx}\right)}$ or one half as large as that for the approach of the leading edge to the sonic point. It is clear from figure 2 that the formation of shocks from compression waves is a faster process and a shock is formed in a finite time.

Application to the Formation of a Normal Shock in a de Laval Nozzle

The theory just developed can be applied to trace the formation of shock waves in a de Laval nozzle. Consider the converging-diverging nozzle shown schematically in figure 5. Suppose the back pressure to be adjusted to such a value that the local speed of sound is reached at the throat but not exceeded. The form of velocity distribution that would be obtained is also shown schematically in figure 5. The small curved region near the local speed of sound in the velocity curve which would be obtained in a practical case has been neglected to simplify the argument.

It is known from many experimental results that if the back pressure is lowered further, a normal shock will form in the nozzle. The formation of this shock can be readily described by use of the foregoing theory if the velocity at the end of the nozzle is sufficiently close to the speed of sound. It will be assumed that the back pressure will be lowered continuously for a time and then held constant. (It will also be assumed that the back pressure is not lowered enough to produce supersonic flow at the end of the nozzle.) While the back pressure is being lowered, the velocity at the end of the nozzle will increase. The space derivative of the velocity disturbance created can be found from equation (16) if downstream-moving disturbances are neglected. Solving this equation for $\frac{\partial u'}{\partial x} = \frac{1}{2}\frac{dQ}{dx}$ yields

$$\frac{\partial u'}{\partial x} = \frac{-\frac{6}{5}u'\frac{du_o}{dx}}{u_o - a_o + \frac{6}{5}u'} - \frac{\frac{\partial u'}{\partial t}}{u_o - a_o + \frac{6}{5}u'} \tag{25}$$

It can be seen from equation (25) that some time before the expansion stops, $\frac{\partial u'}{\partial x}$ will become negative. This result will occur when

$$\frac{6}{5}u'\frac{du_o}{dx} + \frac{\partial u'}{\partial t} < 0$$

since the denominators in equation (25) are negative for subsonic velocities.

A negative value of $\frac{\partial u'}{\partial x}$ means that a compression wave has been produced in the process of decreasing the back pressure to a new steady value. (It should be remarked that no such compression waves are started in the absence of a decelerating velocity field.) This compression wave will travel upstream and will steepen as has been shown previously (equation (23)) to form a compression shock. The compression shock and the wedge-shaped area ahead of it will grow by the addition of expansion pulse area coming from the rear of the nozzle (pulse area is continually contributed since $Q \neq 0$ at the rear). The shock will grow until its intensity and its dissipation are large enough to reach a condition where the nozzle flow is again at equilibrium.

The calculations on which figure 5 is based are given in appendix C. The leading phase which originated while the nozzle back pressure was falling can be calculated from equation (23) and the known propagation velocity of its leading edge. The steady phase which originates after the nozzle back pressure has again become steady can be found by integrating equation (16), use being made of the fact that $\frac{\partial u'}{\partial t} = 0$. for this phase. For the case considered, an infinite decelerating velocity gradient appears at the point where the velocity first reaches the local speed of sound. The shock that follows is calculated step by step from equation (19). The calculations used in plotting figure 5 are thus only a first approximation to the shock motion. A more accurate shock velocity calculation (by the methods of part II) would show the shock to settle at an equilibrium position.

If the unsteady aspect of a channel-flow problem is considered, it is possible to see how shocks arise in the transonic flow. It is quite possible that an analysis of this kind could be extended to two or three dimensions and to the flow over obstacles and that if these extensions were made, the problem of the formation of normal shocks in these more complicated cases could be greatly clarified.

II - THE MOTION AND STABILITY OF SHOCK WAVES IN A CHANNEL FLOW

Preliminary Considerations

The trapping of pulses and the formation of shock waves in decelerating channel flows have been considered in part I. In the case of the central problem, the stability of smooth transonic deceleration, these calculations indicate that a trapped pulse converts a part of the channel flow to the accelerating steady flow. The trapped pulses always involve shock waves and if the approximations of appendix B are used for the shock velocity and if downstream waves created by the shocks are neglected, it is found that the entire pulse becomes stationary in the sonic region of the channel.

The occurrence of these trapped pulses in transonic decelerating channel flow means that, to the order of accuracy of these calculations, this flow has a kind of neutral stability to pulses coming from the rear of the channel. Therefore, it will be necessary to make more accurate shock-velocity calculations before it can be decided whether this flow is stable or unstable. In this part of the paper, the motion of shock waves will be considered more accurately.

It will be assumed that conditions at the ends of the channel are steady; that is, the shock velocity will be calculated for the case where the channel flow is out of equilibrium (for example, where a channel shock has been displaced by a pulse) but where no new disturbances are originating at the end of the channel. Thus, disturbances will be treated by the methods of part I and these more cumbersome calculations will be used to solve stability problems which cannot be solved by the methods of part I.

The flow on the upstream side of the shock will not be related in any way to the shock position or motion because any disturbances created by the shock will not move upstream as rapidly as the shock does. It is therefore possible to calculate the state of the air at all points ahead of the shock wave before the shock motion is determined. For all the applications of the shock-velocity calculations that are contemplated in this paper, an equilibrium flow (either the original equilibrium flow or the alternate one inferred to be produced by a trapped pulse) will be assumed upstream of the shock wave.

In steady flow, once the conditions ahead of the shock are known, it is possible to specify immediately from the Rankine-Hugoniot relations the conditions behind the shock. In unsteady flow, however, since the shock velocity adds another unknown to the problem, it will be necessary to have another relation in addition to the Rankine-Hugoniot relations before the shock velocity and the conditions behind the shock can be specified. Since the conditions upstream from the shock wave are already fully determined, this additional information will be a relation between the variables of condition downstream from the shock. The nature of this relation can be brought out most simply by considering a particular case, namely, the case of a channel discharging into the atmosphere. It will be accurate enough to say that discharging into the constant-pressure atmosphere is equivalent to a constant pressure boundary condition at the end of the nozzle. If a P disturbance moves downstream in the channel, it will of course, involve pressure changes and, when the end of the nozzle is reached, this disturbance will be reflected as an upstream-moving disturbance. Thus, it will be possible for a given P disturbance to calculate the Q disturbance at the end of the nozzle. Now, applying the isentropic wave propagation equations permits the extension of this information to regions upstream. Thus, in general, if the downstream-moving disturbances at a point are known and the boundary conditions at the end of a channel are specified, it would be possible to calculate the upstream-moving disturbances at the same point in the channel. More accurately stated, the downstream-moving disturbances for a previous interval must be known in order to calculate the upstream-moving disturbances.

These problems could probably be treated accurately by the method of characteristics and calculations of this kind would certainly be desirable. However, for the exploratory purposes of this paper, an approximation will be introduced to make it possible to reduce these calculations to a closed form. Thus, the reflection condition will be applied immediately behind the shock instead of at the end of the channel. For example, in treating the constant-pressure end condition, the pressure immediately behind the shock will be assumed to be the subsonic equilibrium pressure appropriate to the shock position. The reflection condition, in combination with the conservation laws, may then be used to obtain an algebraic expression for the shock velocities. This procedure introduces the following errors: First, the changes in the strength of the waves in moving downstream to the channel end and in moving upstream to the shock are neglected. Second, the time required for a wave to move from the shock to the channel end and for the reflection to move back to the shock has been neglected. In cases where the channel following the shock is not too long, this effect will not be serious for shock Mach numbers close to 1, since the shock velocities found are small compared with the local propagation velocities.

Reflection Conditions

The analysis will be made for three types of reflection conditions: constant velocity at the channel end, no reflections, and constant pressure. These cases will provide a considerable variety of reflections; thus a method is provided for estimating the significance of reflections in these problems and, hence, the significance of the two types of errors just mentioned. A further discussion of these errors will be given after the shock-velocity calculations have been presented.

Constant-velocity channel exit.- Constant velocity at the channel exit could be approached practically by arranging a channel to discharge into a machine which takes in a constant volume per unit time. If, in such a setup, a disturbance moves down the channel from the shock wave, it will be reflected from the end of the channel in much the same way as a sound wave is reflected from a closed organ-pipe end. The velocity will thus remain constant at the channel end and, in this case, $u' = 0$ at the end of the channel and $P = -Q$ at that point. As discussed previously, this boundary condition will actually be applied immediately behind the shock.

The shock relation is introduced now in the form

$$
\begin{aligned}
V_1V_2 &= \frac{V_1^{\,2}}{6} + \frac{5}{6}\,a_1^{\,2} \\
&= \frac{V_2^{\,2}}{6} + \frac{5}{6}\,a_2^{\,2}
\end{aligned}
\tag{26}
$$

where V_1 and a_1 are the velocity and velocity of sound immediately upstream of the shock in coordinates fixed with respect to the shock wave and V_2 and a_2 are the velocity and velocity of sound immediately following the shock in these coordinates. The notation to be used in connection with problems of the motion of shock waves is illustrated in figure 6. Returning to a coordinate system fixed with respect to the channel yields for equation (26)

$$6\left(u_1 - U\right)\left(u_2 - U\right) = \left(u_1 - U\right)^2 + 5a_1^{\,2}$$

$$= \left(u_2 - U\right)^2 + 5a_2^{\,2} \tag{27}$$

The constant-velocity reflection condition yields the information $u'_2 = 0$ and hence $u_2 = u_{2_\xi}$. Using this information and eliminating a_2 between the equation (27) gives the following solution for the shock velocity.

$$U = \frac{1}{5}\left[2u_{1_\xi} + 3u_{2_\xi} \pm \sqrt{\left(2u_{1_\xi} + 3u_{2_\xi}\right)^2 - 30u_{1_\xi}u_{2_\xi} + 25a_s^{\,2}}\right] \tag{28}$$

It should be pointed out that the plus sign corresponds to the expansion "shock" which violates the second law of thermodynamics and, therefore, the minus sign must be used.

Nonreflecting channel exit.- The condition where no waves are reflected from the rear of the channel could be approached in the case where the region of the channel containing the shock wave under consideration is followed by a sonic throat and a supersonic region. In this case, the only reflections present would be those produced by the variations in channel area between the shock and the throat. If all the Mach numbers are close to unity, the variation in channel area would be small and, therefore, the reflections should be small.

The absence of reflections means that no upstream-moving disturbance will exist behind the shock in the absence of external disturbances. The reflection condition in this case is then $Q_2 = 0$. Applying this relation immediately behind the shock gives $Q_2 = u'_2 - 5a'_2 = 0$. Using this information in equations (27) makes it again possible to eliminate u'_2 and a'_2 and to obtain

$$U^4 + AU^3 + BU^2 + CU + D = 0 \tag{29}$$

where

$$A = 4\left(\frac{1}{10} u_{2_\xi} - \frac{11}{10} u_{1_\xi} - \frac{1}{2} a_{2_\xi}\right)$$

$$B = 6\left(\frac{24}{25} u_{1_\xi}^{\;2} + \frac{4}{25} u_{2_\xi}^{\;2} - \frac{3}{25} u_{1_\xi} u_{2_\xi} + \frac{3}{5} u_{1_\xi} a_{2_\xi} + \frac{2}{5} u_{2_\xi} a_{2_\xi}\right)$$

$$C = 4\left(\frac{2}{25} u_{1_\xi}^{\;2} u_{2_\xi} - \frac{2}{5} u_{1_\xi}^{\;2} a_{2_\xi} - \frac{3}{5} u_{1_\xi}^{\;3} - \frac{12}{25} u_{1_\xi} u_{2_\xi}^{\;2}\right.$$
$$\left. - \frac{6}{5} u_{1_\xi} u_{2_\xi} a_{2_\xi} - \frac{1}{10} u_{2_\xi} a_s^{\;2} + \frac{1}{10} u_{1_\xi} a_s^{\;2} + \frac{1}{2} a_{2_\xi} a_s^{\;2}\right)$$

$$D = \left(\frac{24}{25} u_{1_\xi}^{\;2} u_{2_\xi}^{\;2} + \frac{12}{5} u_{1_\xi}^{\;2} u_{2_\xi} a_{2_\xi} - a_s^{\;4} + \frac{2}{5} u_{1_\xi} u_{2_\xi} a_s^{\;2} - 2u_{1_\xi} a_{2_\xi} a_s^{\;2}\right)$$

Here again, it is possible to solve for U if u_{1_ξ} and a_{1_ξ} are known. Of the four roots of equation (29), two are imaginary and one corresponds to the expansion "shock" so that only one has physical significance.

<u>Constant-pressure channel exit</u>.- The third boundary condition which will be considered is the case of constant-pressure channel exit. This condition could be approached in a channel that discharges into the atmosphere. Transferring this boundary condition from the channel end to the shock position gives the information that the pressure behind the shock is the same as the pressure which would exist at that point in the steady flow. The constant-pressure boundary condition can thus be stated as $p_2 = p_{2_\xi}$. Using the shock relation

$$\frac{p_2}{p_1} = \frac{7}{6} \frac{(u_1 - U)^2}{a_1^{\;2}} - \frac{1}{6} \tag{30}$$

gives an immediate solution for the shock velocity.

The Stability of Smooth Deceleration through the Speed of Sound

In the case of a channel flow involving smooth deceleration through the speed of sound, it was found in part I that a pulse coming from the rear of the channel was trapped in the sonic region of the channel. This result means that, to the order of accuracy of these calculations, the flow was neutrally stable to pulses coming from the rear and that more accurate calculations are needed to decide whether this flow is stable or unstable.

Two kinds of approximations were made in deriving the trapped-pulse result. First, the reflections produced by isentropic waves and the terms containing $M_o^2 - 1$ as a factor in equations (14) and (15) were neglected. The expansion waves were found to replace a part of the original decelerating equilibrium flow with a flow close to the equilibrium accelerating flow which is also possible for this channel. (See fig. 4.) It was then inferred that exact calculations would yield the exact accelerating equilibrium flow. If this inference is correct, the exact final state of an expansion wave has been obtained by this method so that the approximations introduced in the calculations of the expansion waves will have no effect on the final results.

Second, rough approximations to the shock velocity developed in appendix B were used and, with these approximations, it was found that the shock neither consumed the pulse nor made it grow. Hence, any change in the shock velocity from these approximate values would lead to consumption or growth of the trapped pulses and, thus, the more accurate shock-velocity formulas just derived can be used to determine the stability of smooth deceleration through the speed of sound.

For the case where the equilibrium flow is smooth deceleration through the speed of sound, the quantities appearing in equations (28) through (30) can readily be evaluated and the shock velocity calculated. It will be seen from these equations that the shock velocity depends only on local conditions at the shock position. These local conditions, of course, are all functions of a single parameter and the parameter used to present these results is the Mach number immediately ahead of the shock.

Results found for the three reflection conditions are plotted in figure 7. It will be seen that the shock velocity is always negative; that is, the shock always moves upstream. Thus, in the case of the trapped expansion pulse (fig. 3), the shock will move upstream and eventually will consume the pulse. Smooth deceleration through the speed of sound is therefore stable to expansion pulses coming from the rear. On the other hand, in the case of the trapped compression pulse, the shock wave will again move upstream, and in this case, the pulse will continuously grow (at least as long as the shock wave is in the converging part of the channel). Thus, smooth deceleration through the speed of sound is unstable to compression pulses coming from the rear.

Stability of Shock Position in Channel Flows

The previous discussion shows that, in stable channel flows, deceleration through the speed of sound will be accomplished in a shock wave. Consider now a converging-diverging channel: There are, in general, two possible shock positions which will yield equilibrium flows; that is, the shock can be either in the converging or diverging part of the channel. Experimentally, the shock is found to be stable in the diverging part of the channel and unstable in the converging part of the channel. It will be interesting to apply the formulas just developed to examine this problem theoretically.

A more precise statement of the problem which will be studied is as follows: Consider a shock wave in a channel flow. Consider that a small pulse has displaced the shock wave by a short distance ξ. Since only small disturbances will be considered, it will be convenient to use a linearized form of the shock-velocity equations. If now, the quantities u, a, et cetera are expanded in Taylor series in ξ about the equilibrium shock position, only the first-power terms should be retained to obtain a linearized equation. Furthermore, since the disturbances are small, the shock velocity can be assumed to be small compared with the local velocities or velocities of sound and higher powers of the shock velocity can be neglected. Performing these operations permits equations (28) through (30) to be written in the form

$$\frac{d\xi}{dt} + \tau\xi = 0 \qquad (31)$$

where $\frac{d\xi}{dt}$ is identical with U and the relaxation time τ for the various reflection conditions is as follows:

Constant velocity:

$$\tau \frac{du_{1_\xi}}{dx} = \frac{2u_{1_o} + 3u_{2_o}}{\frac{7}{2} u_{2_o}\left(M_{1_o}{}^2 - 1\right)} \tag{32}$$

No reflections:

$$\tau \frac{du_{1_\xi}}{dx} = \frac{3U_{1_o}{}^2 - \frac{5}{6} a_s{}^2 + 3u_{2_o}{}^2 + a_{2_o}\left(2u_{1_o} + 3u_{2_o}\right)}{\frac{7}{2} u_{2_o}\left(M_{1_o}{}^2 - 1\right)\left(u_{2_o} + a_{2_o}\right)} \tag{33}$$

Constant pressure:

$$\tau \frac{du_{1_\xi}}{dx} = \frac{-10}{\left(M_{1_o}{}^2 - 1\right)\left(\frac{11 - 5M_{1_o}{}^2}{M_{1_o}{}^2 - 1} + \frac{\frac{7u_{2_o}{}^2}{a_{1_o}{}^2} - \frac{M_{2_o}{}^2}{M_{1_o}{}^2}}{M_{2_o}{}^2 - 1}\right)} \tag{34}$$

Values of $\tau \frac{du_{1_\xi}}{dx}$ are plotted in figure 8 against the equilibrium Mach number ahead of the shock. Figure 8 shows that $\tau \frac{du_{1_\xi}}{dx}$ is always positive and the shock position is therefore stable (positive τ) when $\frac{du_{1_\xi}}{dx}$ is positive and unstable when $\frac{du_{1_\xi}}{dx}$ is negative. The shock position is thus stable in diverging channels and unstable in converging channels in agreement with experiment.

GENERAL DISCUSSION

Discussion of Simplifying Assumptions

It should be pointed out again that the best way to indicate the effects of the simplifying assumptions introduced in this paper would be to repeat some of the calculations made herein by the method of characteristics and to compare the results obtained with those obtained in this paper. Since such calculations are not yet available, however, it would be desirable to see what information can be obtained from the results themselves in order to define their range of validity. This discussion is given with the central problem - stability of smooth transonic deceleration - in mind, but it will be clear that similar conclusions could be reached for cases where the equilibrium flow involves a shock.

The assumptions of part I led to the trapped-pulse result and to the inference that expansion waves replace the original uniformly decelerating flow with the steady accelerating flow appropriate to the channel. These assumptions have already been discussed in the section on the stability of smooth deceleration through the speed of sound, and it has been pointed out that if this inference is adopted, errors due to the assumptions introduced to facilitate the calculations of isentropic wave propagation in part I are eliminated. The shock velocity formulas of appendix B are, however, only rough approximations.

In part II, more accurate formulas for the velocity of the shock were found, which indicated that the trapped expansion pulses are consumed by the shock and that the trapped compression pulses continually grow as a result of the shock motion. The history of a pulse has thus been divided into two intervals. During the first interval, the consumption of the pulse by the shock has been neglected. During the second interval, it has been assumed that the pulse is fully trapped. There will be, of course, an intermediate interval during which the rates of the two processes are comparable. If the order of magnitude of the times in which a pulse is consumed by the shock is much larger than the times that a pulse requires to approach the trapped state, the intermediate interval will not be very important. The importance of this intermediate interval can therefore be clarified by calculating the fraction of the pulse consumed by the shock motion in the time required for the trapping processes which is of the order of $\dfrac{1}{-\dfrac{du_0}{dx}}$. Calculations of this kind are presented in figure 9.

It can be concluded from this figure that the intermediate interval does not play an important role so long as the pulse area of the initial pulse is not large enough so that high Mach number shocks are involved.

A third group of errors was introduced in part II by applying the reflection conditions after the shock instead of at the end of the channel. The significance of this approximation can be estimated by comparing the results obtained in figures 7 and 8 for the various reflection conditions. The changes in reflection condition considered are seen to produce only a small change in the results where the shock Mach number is close to 1. Therefore, it seems likely that the results would not be greatly altered if high Mach number shocks are not involved by changes in the strength of disturbances in propagating between the shock and the end of the channel.

The time required for disturbances to propagate between the shock and the end of the channel has also been neglected and the significance of this approximation can be brought out by considering the sequence of events which would occur in the case where a very long channel follows a stable channel shock. Assume that a pulse moves upstream and displaces the shock. The shock then returns to equilibrium at a rate which corresponds to the no-reflections case. A downstream-moving pulse will be given out by the shock while it is out of equilibrium. When this pulse reaches the end of the channel, it will be reflected as an upstream-moving pulse and will now displace the shock in the opposite direction. (See fig. 8, constant-pressure or constant-velocity end conditions.) This process will repeat indefinitely and there is the possibility, which should be investigated, that a divergent oscillation could result in some circumstances. For the cases where this oscillation does not diverge (and it is clear from experiment that these cases are an important group), the application of the boundary conditions immediately behind the shock will suppress the damped oscillatory motion which would be expected for long channels and results in a monotonic return to or divergence from the equilibrium position.

Application to Supersonic Diffusers

An interesting application of the present analysis will be to consider the maximum pulse which a supersonic diffuser flow with a shock in a diverging channel can absorb and yet return to its initial configuration (that is, its initial shock position). Consider first the effect of expansion pulses which will move the shock downstream. As long as the shock is not moved beyond the end of the diverging part of the channel, it would be expected that the stability found earlier would return it to its original position.

The effect of compression pulses strong enough to push the shock through the diffuser throat is more interesting. After the pulse has completely interacted with the shock wave (that is, when the trailing edge of the pulse has just reach the shock), the shock will be displaced by an amount which can be readily estimated from the rule of conservation of pulse area. The results of part I indicate that the flow behind the shock wave will assume the alternate (subsonic) steady flow possible for this channel. If the pulse is sufficiently strong, the displaced position of the shock will be in the converging part of the channel where the shock position is unstable. There is, of course, an unstable equilibrium position for the shock in the converging part of the channel and the displaced shock will move away from this unstable equilibrium position. If now the displaced shock position is downstream from the unstable equilibrium position, it will move further downstream and eventually return to the diverging part I of the channel where it will again assume its stable equilibrium position. On the other hand, if the displaced position is upstream from the unstable equilibrium shock position, it will continue to move upstream and eventually convert the supersonic flow in the converging part of the channel to a subsonic flow. Thus, the supersonic flow with a shock in the diverging part of the channel is stable to compression pulses which are not sufficiently strong to displace the shock beyond the unstable equilibrium position in the converging part of the channel.

In the practical design of supersonic diffusers, it is desirable to ensure that disturbances of a given magnitude (that is, a given pulse area) do not force the shock beyond this limiting position. It is, of course, also desirable that the equilibrium shock intensity be kept as low as possible. It appears, therefore, that diffusers with a long throat region, which produce a velocity distribution such as that shown in figure 10, should be considered. In this case, the compression pulse area that can be absorbed by the diffuser, that is, the area ABCD, is increased without increasing the shock intensity. However, it will be noticed that the skin friction would also be increased in this way. Further investigation will be necessary to determine the optimum throat length and shape.

Suggestions for Future Research

It would be interesting to perform calculations on some of the problems which have been studied herein by the characteristics method. For example, the central problem of this paper, stability of smooth deceleration through the speed of sound, could probably be studied by this method. By starting with equations (14) and (15), it should be possible to trace the propagation of an upstream-moving pulse. After a shock has been formed in the pulse, it would be necessary, of course, to introduce the Rankine-Hougonoit relations to trace further development. It would, in particular, be very valuable to see whether more exact calculations confirm the inference that a trapped pulse converts a portion of a decelerating channel flow to a steady accelerating flow.

An experimental check on part I of this theory could be obtained by the use of high-speed flow measurements. Consider a nozzle in which the flow just reaches but does not exceed the local speed of sound. In such a flow, it would be expected that the trapped pulses (or nearly trapped pulses) would be present due to natural or artificial disturbances coming from the rear. It would be anticipated that the characteristic triangular shape of trapped pulses (which would be expected also if density were plotted instead of velocity) predicted from this theory would be readily observable.

An experimental investigation of supersonic diffusers with long throats should be conducted. This study would be particularly interesting in the case of high Mach number diffusers with variable geometry. When the geometry is variable, the only limitation on the minimum shock intensity would be that adequate stability to disturbances would be required. In many applications, a high disturbance level will be present and an effort should be made in experiments of this kind to simulate the actual disturbances which would be present in the anticipated practical application.

As has been pointed out previously, a study of unsteady transonic flow in two or three dimensions would probably do much to clarify the problems of the formation of normal shocks in these flows.

CONCLUDING REMARKS

The propagation of upstream-moving pulses in the sonic region of a nonviscous decelerating channel flow has been studied. It is pointed out that a fairly complete history of a small pulse can be obtained by considering the two types of processes which occur separately (parts I and II).

In part I, Riemann's theory of the propagation of finite-amplitude disturbances in a homogeneous medium has been extended to the case where upstream-moving pulses are superposed on a decelerating channel flow. It is concluded that a pulse approaches a "trapped" state in which it converts a portion of the channel to the accelerating flow which is an alternate steady-state solution for the channel. Shock waves are formed as the pulse is propagated, as in Riemann's problem. If it is assumed that the intensity of the shocks is very small, then, to a first approximation, it is shown that the shocks neither consume the pulse nor make it grow. It is also shown that when a pulse interacts with a weak shock which forms part of the original equilibrium channel flow, the pulse is transformed into a displacement of the shock wave. These facts can be summed up in the following approximate rule: the quantity $\int Q dx$ (called pulse area) is conserved in the propagation of upstream-moving pulses in the sonic region of a channel flow. The theory is applied to trace the development of a shock in a de Laval nozzle as the back pressure is lowered. It is shown that a shock forms inevitably if the back pressure is lowered below the value which first produces sonic velocities at the throat. It is hoped that this calculation will be useful in leading to the solution of the more difficult problems of the formation of shocks in two- or three-dimensional-flow problems.

The central problem considered in this paper is the stability of channel flows involving smooth (shock-free) deceleration through the speed of sound. For this problem, the analysis of part I indicate that a pulse coming from the rear of the channel will remain permanently trapped in the sonic region. Thus, from part I, the flow has neutral stability to upstream-moving pulses. Therefore, it appears that a more accurate analysis must be made before stability problems can be considered.

In part II, the assumption of weak shocks (used in deriving the rule of conservation of pulse area in part I) is dropped and the motion of shock waves is considered more accurately. It is necessary to assume some type of reflection condition at the downstream channel end before the problem of shock motion is competely specified. It is made plausible from the results that little difference in the shock velocity occurs when a range of reflection conditions are assumed. Therefore, for the exploratory study of stability problems, a simplified treatment of reflections is used which permits the calculations to be made in closed form rather than by the laborious method of characteristics.

By combining the various reflection conditions with the Rankine-Hugoniot relations, the velocity of shock waves in channel flows which are out of equilibrium is computed. For cases where a trapped pulse is superposed on a smooth deceleration through the speed of sound, it is found that the shock always moves upstream. Thus, trapped expansion pulses are consumed by the shock motion and trapped compression pulses inevitably grow. It is concluded, therefore, that smooth transonic decleration is unstable to compression pulses coming from the rear of the channel.

The stability to small disturbances of channel flows which, at equilibrium, involve shock waves is also considered in part II. It is shown that the shock position is unstable in converging channels and is stable to small disturbances in diverging channels. It is made plausible that the disturbance level can affect the minimum shock intensity which can be attained in a practical supersonic diffuser. The theory indicates that supersonic diffusers with long throats may permit a lower shock intensity and thus have a higher efficiency when a high disturbance level is present.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., November 18, 1946

APPENDIX A

SYMBOLS

x distance measured along the axis of the channel; x is taken positive in downstream direction

t time

$T = \frac{du_o}{dx} t$

u velocity in the x direction

a local velocity of sound

ρ local density

γ ratio of heat capacity at constant pressure to heat capacity at constant volume, equal 7/5 for room-temperature air; this value of 7/5 is assumed throughout this report

M_o Mach number (u_o/a_o)

$u' = u - u_o$

$a' = a - a_o$

P $\begin{cases} u + 5a & \text{for section on disturbances superposed on a homogeneous medium} \\ u' + 5a' & \text{for the section on disturbances superposed on a flow field} \end{cases}$

Q $\begin{cases} u - 5a & \text{for section on disturbances superposed on a homogeneous medium} \\ u' - 5a' & \text{for the section on disturbances superposed on a flow field} \end{cases}$

$\frac{\delta}{dt}$ denotes derivation with respect to time when point under observation moves with local propagation velocity

ξ displacement of a shock from its equilibrium position

U velocity of a shock wave in stationary coordinates

V flow velocity in coordinates moving with a shock wave

S local channel area

$$\epsilon = \frac{\partial \xi}{\partial x}$$

$$E = \frac{\epsilon}{4\frac{du_o}{dx}}$$

Subscripts:

s stagnation conditions

o local equilibrium flow conditions

cr equilibrium conditions for $M = 1$

1 conditions ahead of a shock

2 conditions behind a shock

1_o equilibrium flow condition upstream of a shock at the equilibrium shock position

2_o equilibrium flow condition downstream of a shock at the equilibrium shock position

1_ξ equilibrium supersonic flow condition upstream of a shock calculated at the displaced shock position

2_ξ equilibrium subsonic flow condition downstream of a shock calculated at the displaced shock position

Defined in figure 6

A bar over a quantity denotes a lower limit of integration

APPENDIX B

APPROXIMATE FORMULAS FOR THE VELOCITY OF PROPAGATION OF SHOCK WAVES FOR USE IN PART I

The exact computation of shock velocity, in general, involves cumbersome algebra; therefore, for use in part I, approximate expressions will be developed for the shock velocity. First, consider the case where an upstream-moving disturbance involving a shock is superposed on a smooth steady flow u_o. Conservation of energy across the shock can be written

$$V_1^2 + 5a_1^2 = V_2^2 + 5a_2^2 \tag{B1}$$

where the velocities are measured relative to a coordinate system moving with the shock velocity U. Returning to the former convention of measuring velocities relative to stationary coordinates gives, for equation (B1)

$$(u_1 - U)^2 + 5a_1^2 = (u_2 - U)^2 + 5a_2^2 \tag{B2}$$

Introducing the steady flow and disturbance quantities, as before, gives

$$u_o^2 + 2u'_1u_o + u'^2_1 - 2(u_o + u'_1)U + U^2 + 5a_o^2 + 10a_oa'_1 + 5a'^2_1$$

$$= u_o^2 + 2u'_2u_o + u'^2_2 - 2(u_o + u'_2)U + U^2 + 5a_o^2 + 10a_oa'_2 + 5a'^2_2 \tag{B3}$$

If only upstream-moving disturbances are present, $P = 0$, $u' = -5a'$, and equation (B3) can be simplified to

$$U = u_o - a_o + \frac{1}{2}\left[\frac{6}{5}(u'_1 + u'_2)\right] = u_o - a_o + \frac{3}{10}(Q_1 + Q_2) \tag{B4}$$

In this calculation downstream-moving disturbances created by the shock itself have been neglected.

The case where a steady flow involves a shock and where small upstream-moving disturbances interact with the shock also yields a simple result. In this case, the steady flow velocities on the two sides of the shock are different and will be denoted by u_{1_ξ} and u_{2_ξ}. (See fig. 6.) Equation (B2) can now be written

$$u_{1_\xi}^{\ 2} + 2u'_1 u_{1_\xi} + u'^{\,2}_1 - 2\left(u_{1_\xi} + u'_1\right)U + U^2 + 5a_{1_\xi}^{\ 2} + 10a_{1_\xi}a'_1 + 5a'^{\,2}_1$$

$$= u_{2_\xi}^{\ 2} + 2u'_2 u_{2_\xi} + u'^{\,2}_2 - 2\left(u_{2_\xi} + u'_2\right)U + U^2 + 5a_{2_\xi}^{\ 2} + 10a_{2_\xi}a'_2 + 5a'^{\,2}_2 \qquad \text{(B5)}$$

Since the disturbances are now assumed small compared with the corresponding steady flow quantities, the shock velocity will also be small compared with the steady-flow velocity. If second-order small quantities are neglected and it is noted that $u_{1_\xi}^{\ 2} + 5a_{1_\xi}^{\ 2} = u_{2_\xi}^{\ 2} + 5a_{2_\xi}^{\ 2}$ equation (B5) can be written

$$U = \frac{u_{2_\xi}u'_2 - u_{1_\xi}u'_1 + 5a_{2_\xi}a'_2 - 5a_{1_\xi}a'_1}{u_{2_\xi} - u_{1_\xi}} \qquad \text{(B6)}$$

If only upstream-moving disturbances are assumed to be present, $u'_1 = -5a'_1$ and $u'_2 = -5a'_2$. Equation (B6) can then be written

$$U = \frac{Q_2}{2}\left(\frac{u_{2_\xi} - a_{2_\xi}}{u_{2_\xi} - u_{1_\xi}}\right) + \frac{Q_1}{2}\left(\frac{u_{1_\xi} - a_{1_\xi}}{u_{1_\xi} - u_{2_\xi}}\right) \qquad \text{(B7)}$$

If the variation of the steady flow velocities with ξ is neglected, equation (B7) can be written

$$U = \frac{Q_2}{2}\left(\frac{u_{2_o} - a_{2_o}}{u_{2_o} - u_{1_o}}\right) + \frac{Q_1}{2}\left(\frac{u_{1_o} - a_{1_o}}{u_{1_o} - u_{2_o}}\right) \tag{B8}$$

Thus, in going from equation (B7) to equation (B8) effectively, the effects due to stability of the shock wave have been neglected in comparison with the disturbance effects.

A limiting value of the quantities in parenthesis in equation (B8) as all the velocities approach the local speed of sound a_{cr} can be found by differentiating the steady-flow relationship

$${u_{2_o}}^2 + 5{a_{2_o}}^2 = 5{a_s}^2$$

Thus, there is obtained, if $\Delta u_{2_o} = u_{2_o} - a_{cr}$ and $\Delta a_{2_o} = a_{2_o} - a_{cr}$.

$$a_{cr}\,\Delta u_{2_o} + 5a_{cr}\,\Delta a_{2_o} = 0$$

or

$$\Delta a_{2_o} = -\frac{\Delta u_{1_o}}{5}$$

Similarly, differentiating the shock relation

$$u_{1_o}u_{2_o} = \frac{5}{6}\,{a_s}^2$$

gives

$$\Delta u_{1_o} = -\Delta u_{2_o}$$

From these first approximations, it is found that, for values of M close to 1

$$\frac{u_{2_o} - a_{2_o}}{u_{1_o} - u_{2_o}} = -\frac{3}{5}$$

and, similarly,

$$\frac{u_{1_o} - a_{1_o}}{u_{1_o} - u_{2_o}} = + \frac{3}{5}$$

Thus, equation (B8) becomes

$$U = \frac{3}{10}(Q_1 + Q_2) \tag{B9}$$

APPENDIX C

CALCULATIONS OF THE FLOW SHOWN IN FIGURE 5

For purposes of discussion, the disturbances will be divided into three phases: (1) a leading phase which originates while the pressure at the end of the nozzle is falling; (2) a steady phase which originates after the nozzle back pressure has reached a steady value; and (3) a shock phase which appears, in the case assumed, at the juncture of the two previous phases.

In order to simplify the calculations of the leading phase, the rate of pressure drop (or velocity increase) at the nozzle end was chosen to make ϵ or $\frac{\partial u'}{\partial x}$ constant. The leading phase thus appears as a straight line and can be seen from equation (23) to remain a straight line as long as the disturbance remains in a uniformly decelerating field. The motion of the leading edge of the disturbance in a short time Δt will be $\Delta t(u_o - a_o)$. The end point of the leading phase can be followed similarly when the steady phase propagation has been calculated.

In the steady phase, no changes are being propagated in either direction; hence, $\frac{\partial u'}{\partial t} = 0$. Thus, a single curve will be an envelope for the disturbances at successive times. This curve can be found by integrating equation (16), while making use of the fact that $\frac{\partial u'}{\partial t} = 0$. Equation (16) can then be written

$$\frac{du'}{dx} = \frac{-\frac{6}{5}\frac{du_o}{dx}u'}{u_o - a_o + \frac{6}{5}u'} \tag{C1}$$

For the nozzle assumed $\frac{du_o}{dx}$ is a constant which will be denoted by $\frac{1}{b}$; then, since M_o is close to 1

$$u_o - a_o = \frac{6}{5}\frac{x}{b}$$

and

$$\frac{du'}{dx} = -\frac{\frac{u'}{x}}{1 + b\frac{u'}{x}}$$

This equation can be written

$$x\,du' + u'\,dx + bu'\,du' = 0$$

and integrated to give (with quantities referring to the lower limit denoted by bars)

$$xu' - x\bar{u}' + \frac{bu'^2}{2} - \frac{b\bar{u}'^2}{2} = 0$$

From this equation, u' can be obtained as a function of x from the known value of u' at the end of the nozzle. This result has been plotted as the envelope in figure 5. It will be noticed that the slope becomes infinite at the point where the steady-phase curve reaches the local speed of sound. This result can be readily verified from equation (C1) where the denominator of the right-hand side vanishes at the local speed of sound. This point, then, is where a shock first forms. The propagation of the disturbance up to the point where a shock forms can now be readily calculated. Once formed, the shock will connect the leading phase with the steady phase.

The shock position and intensity at any time can be calculated step by step as follows: First, the leading phase is determined from the known propagation velocity of its leading edge and its slope as found from equation (23). Second, the shock position is determined by adding to the previous shock position an increment corresponding to the shock velocity (equation (B4)). Third, the shock is then a vertical line (in fig. 5) connecting the leading phase with the steady phase at the known shock position.

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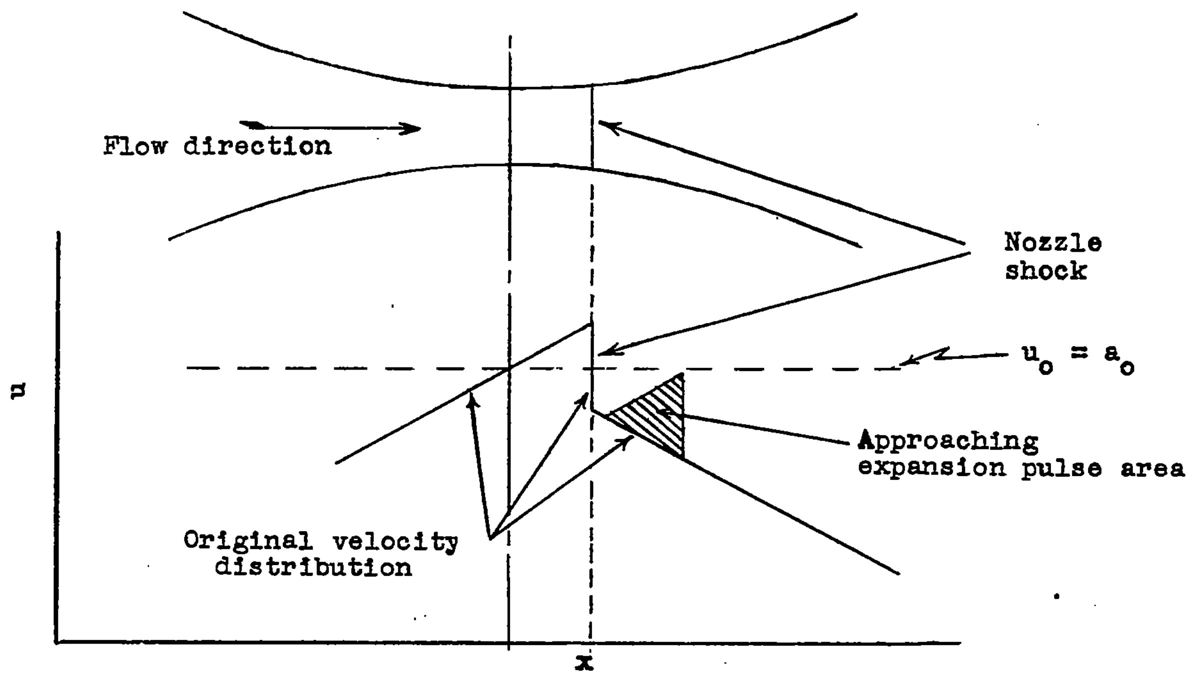
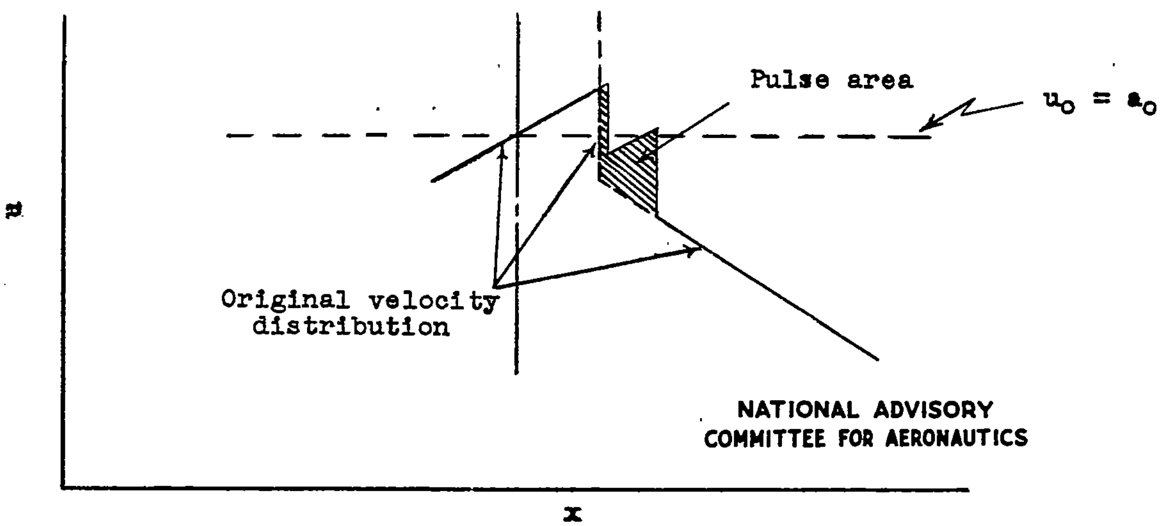


(a) Disturbance approaches shock.



(b) Interaction of disturbance with shock.

Figure 1.- The interaction of an expansion pulse with a nozzle shock. The analysis of part I shows that the pulse area which disappears on the downstream side of the shock is equal to the pulse area which appears due to the shock displacement on the upstream side of the shock. Note that the pulse areas plotted here are $\frac{1}{2}\int Q \, dx$ since $u' = \frac{Q}{2}$.

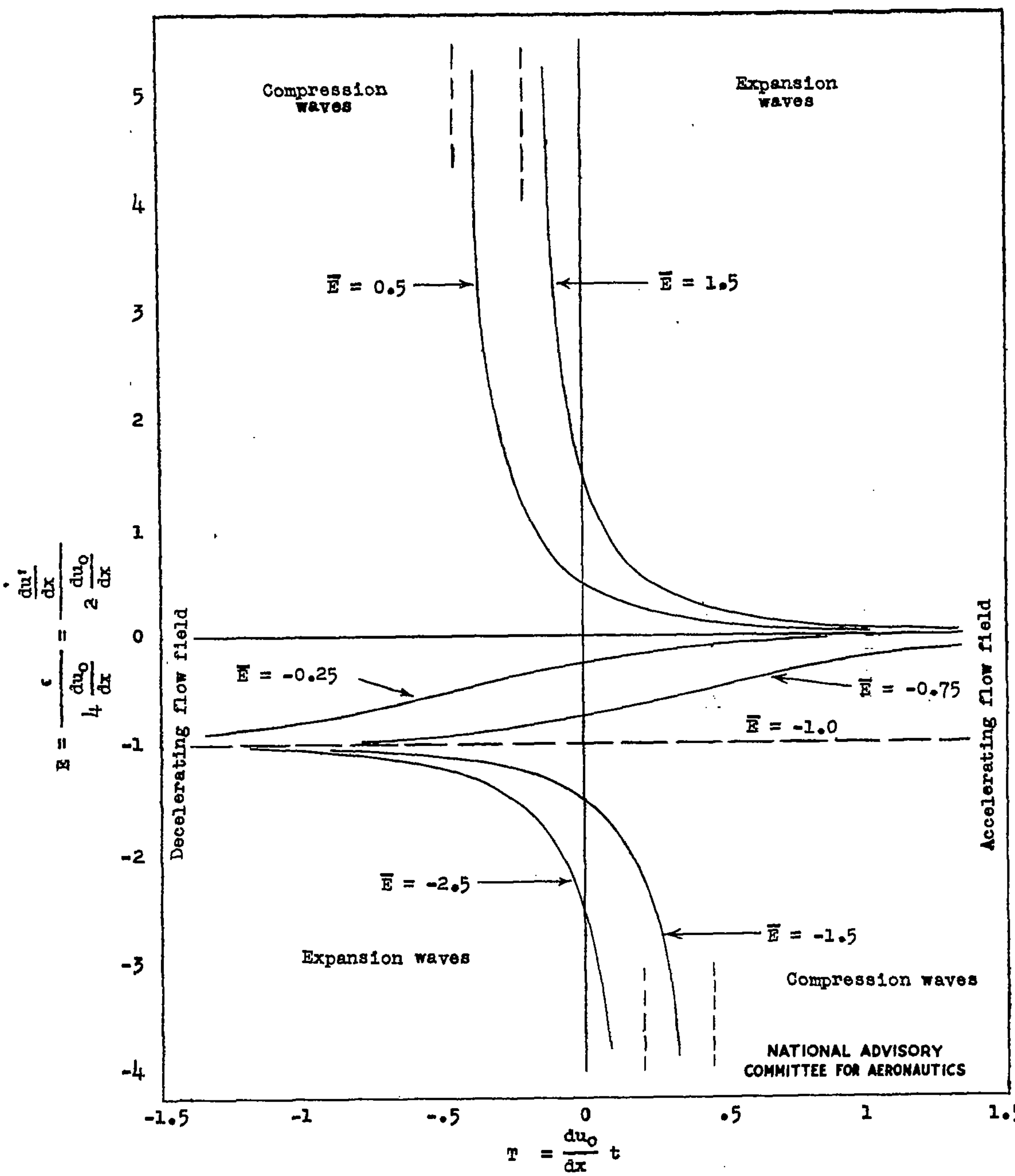


Figure 2.- The propagation of upstream-moving pulses in a uniform sonic region of a channel. Moving to the right for accelerating fields or to the left for decelerating fields, the slope changes of a wave as the point under observation moves with the local propagation velocity are found. In contrast to Riemann's results, in an accelerating flow field, a compression wave must have a certain initial steepness $\bar{E} < -1.0$ before it will steepen to form a shock. Also, in a decelerating field, all expansion waves tend to approach the slope $E = -1.0$ or $\frac{du}{dx} = -\frac{du_o}{dx}$. These effects are illustrated schematically in figure 3 for a decelerating field.

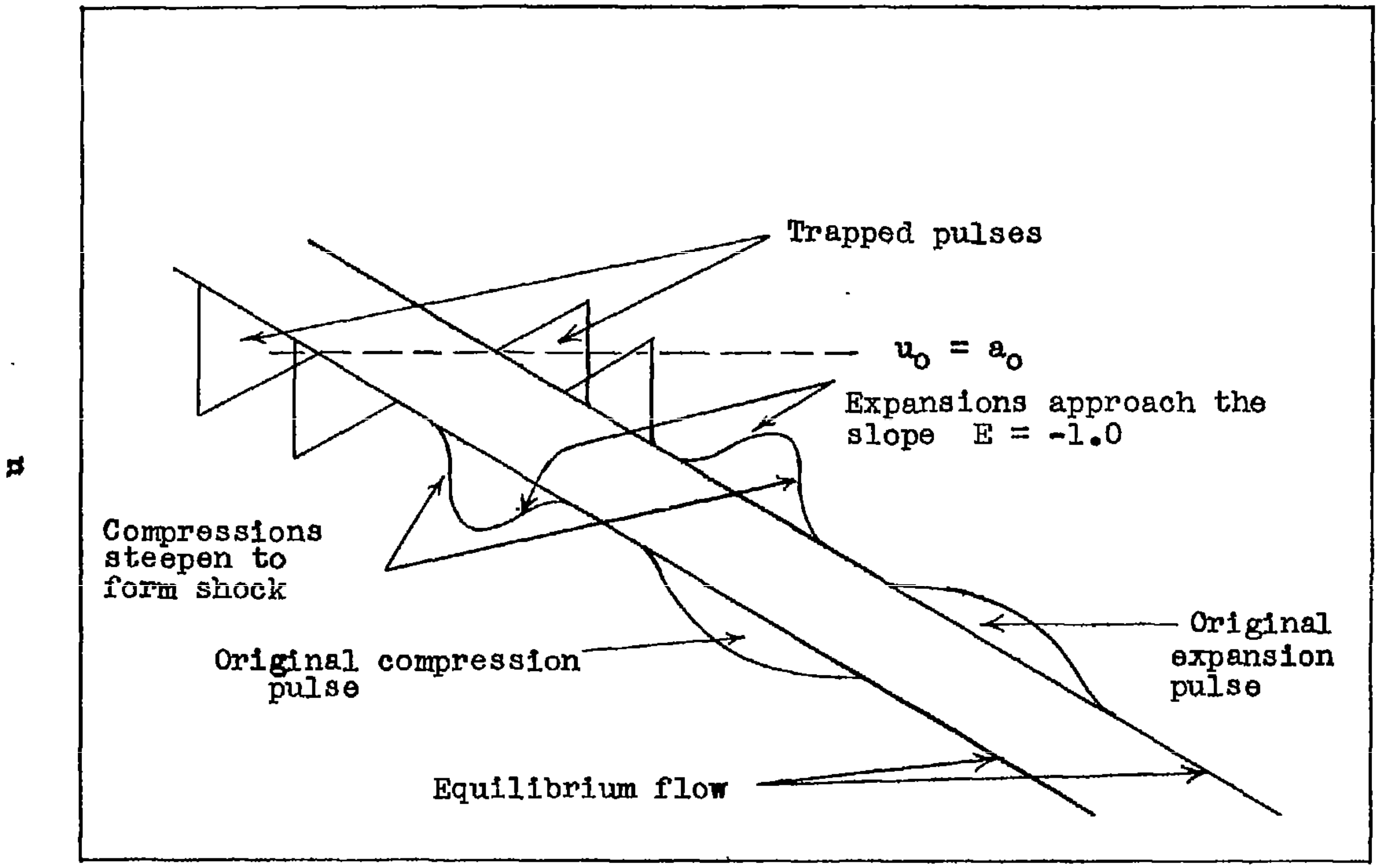


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Figure 3.- Schematic illustration of four successive positions in the progress of expansion and compression pulses in a uniformly decelerating field. The pulse area is conserved and pulse shape approaches an isosceles triangle. When the leading portion of the expansion pulse or the trailing portion of the compression pulse approaches the local velocity of sound, the pulse approaches a "trapped" state.

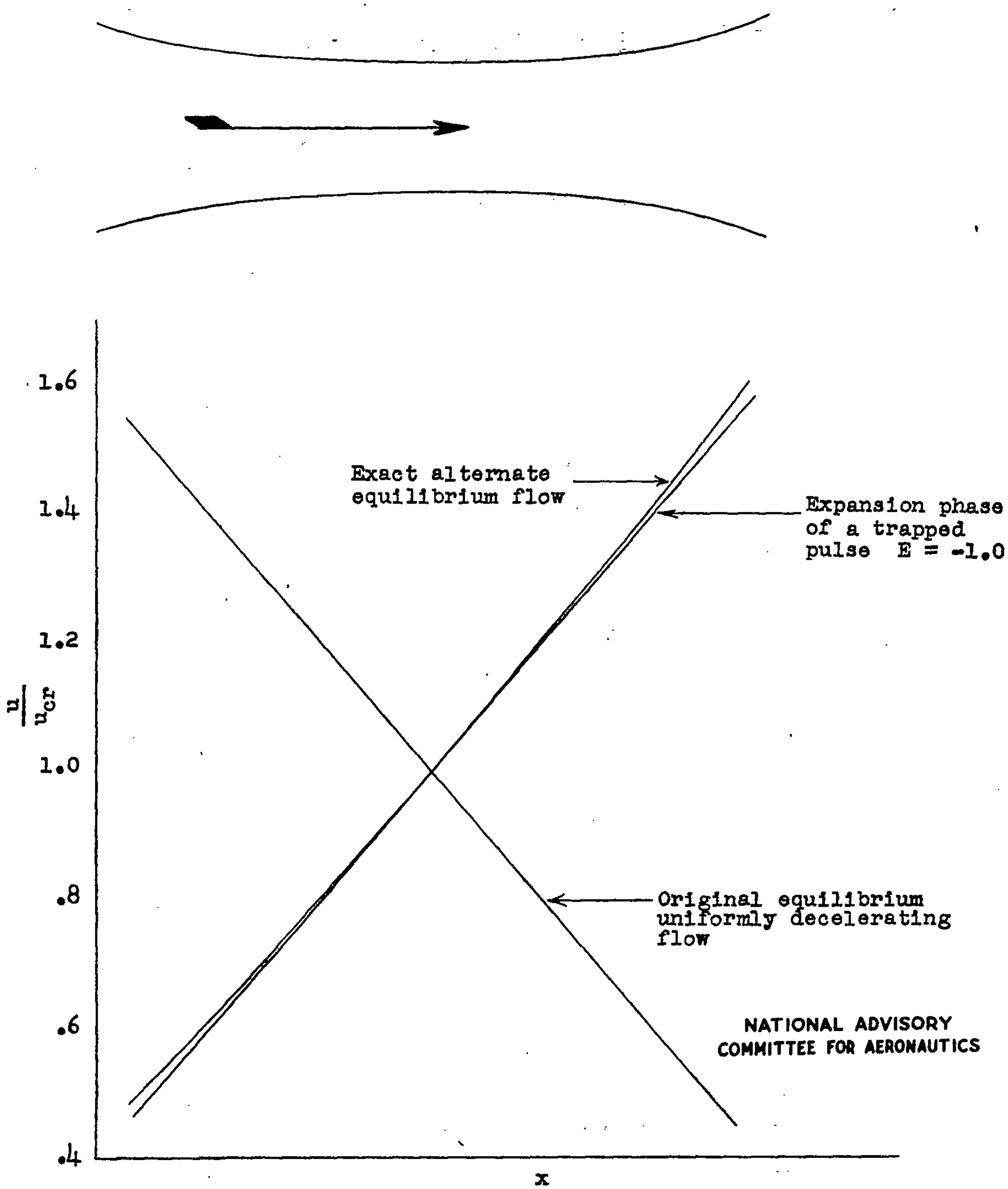

Figure 4.- Comparison of the velocity distribution for the trapped expansion wave with the accelerating equilibrium flow for the same channel.

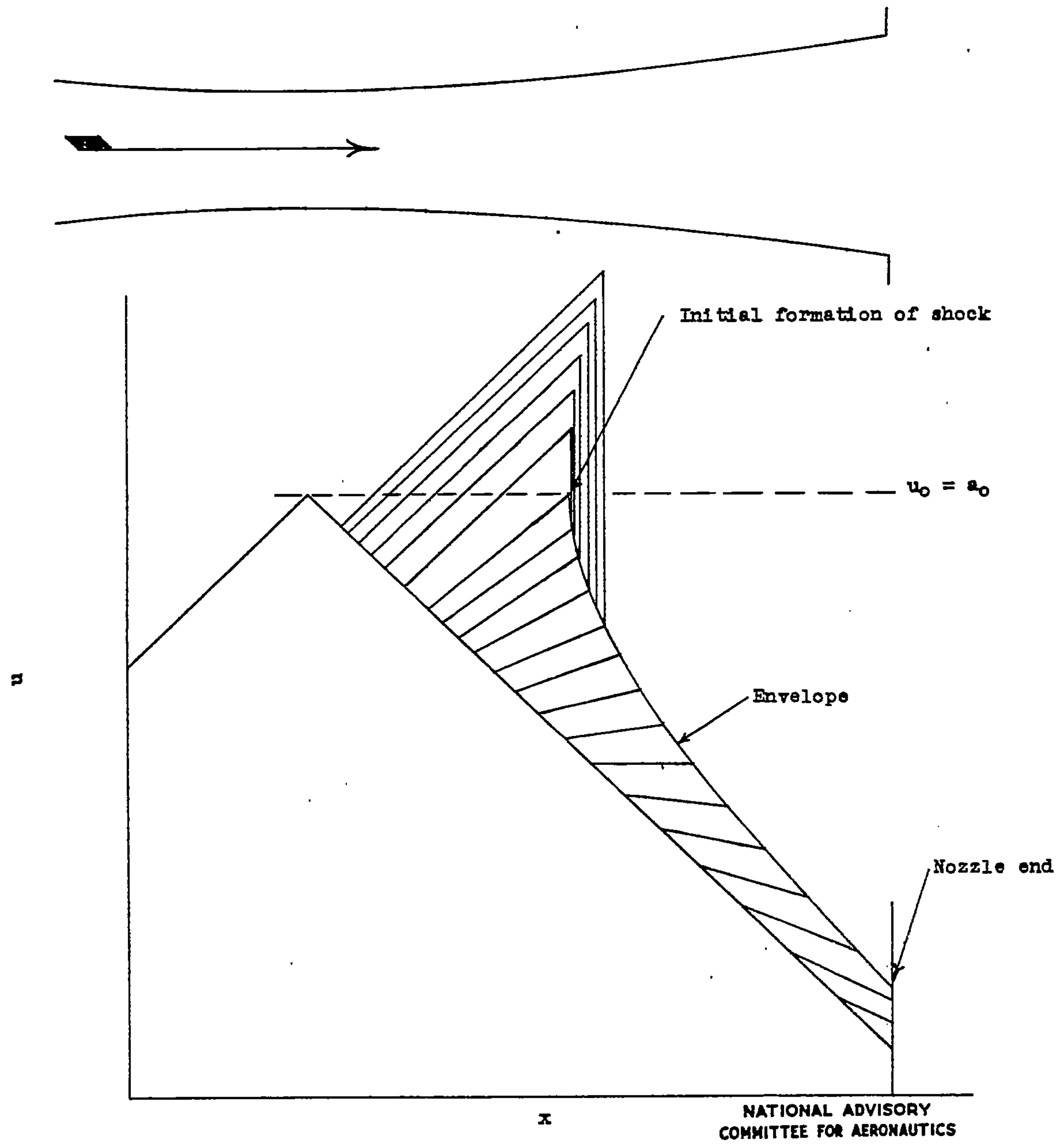

Figure 5.- The development of a normal shock in a de Laval nozzle as the back pressure is lowered at the nozzle end. The envelope curve is calculated in appendix C. The successive expansion wave positions are calculated by finding the propagation velocities of the end points. The time intervals between successive positions are equal up to the formation of shock and double thereafter. The shock velocity is calculated from the approximate equation (B4).

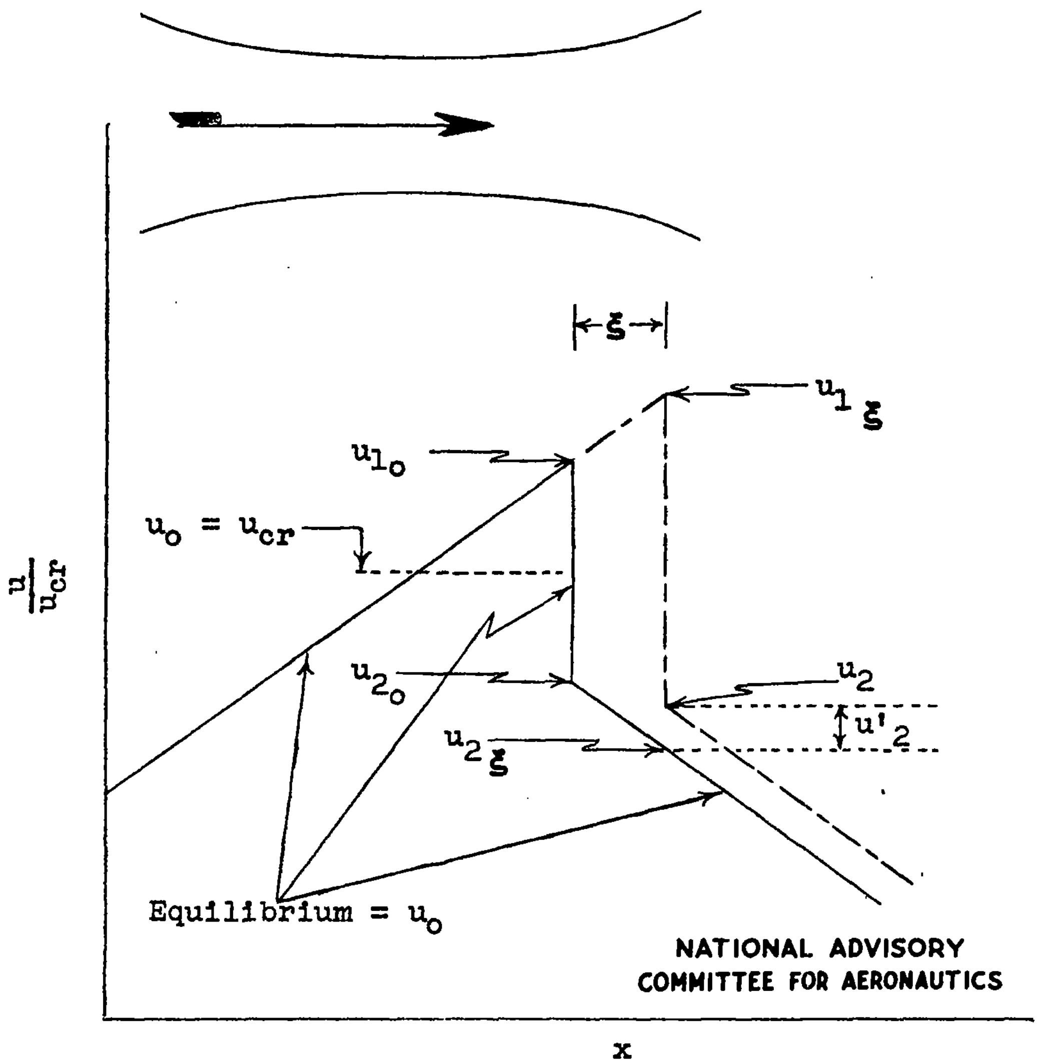

Figure 6.- The notation used in shock-velocity problems. For the case where no initial shock is involved in the equilibrium flow $u_{1_o} = u_{2_o} = u_{cr}$. Explicit definitions of the subscripts are given in appendix A.

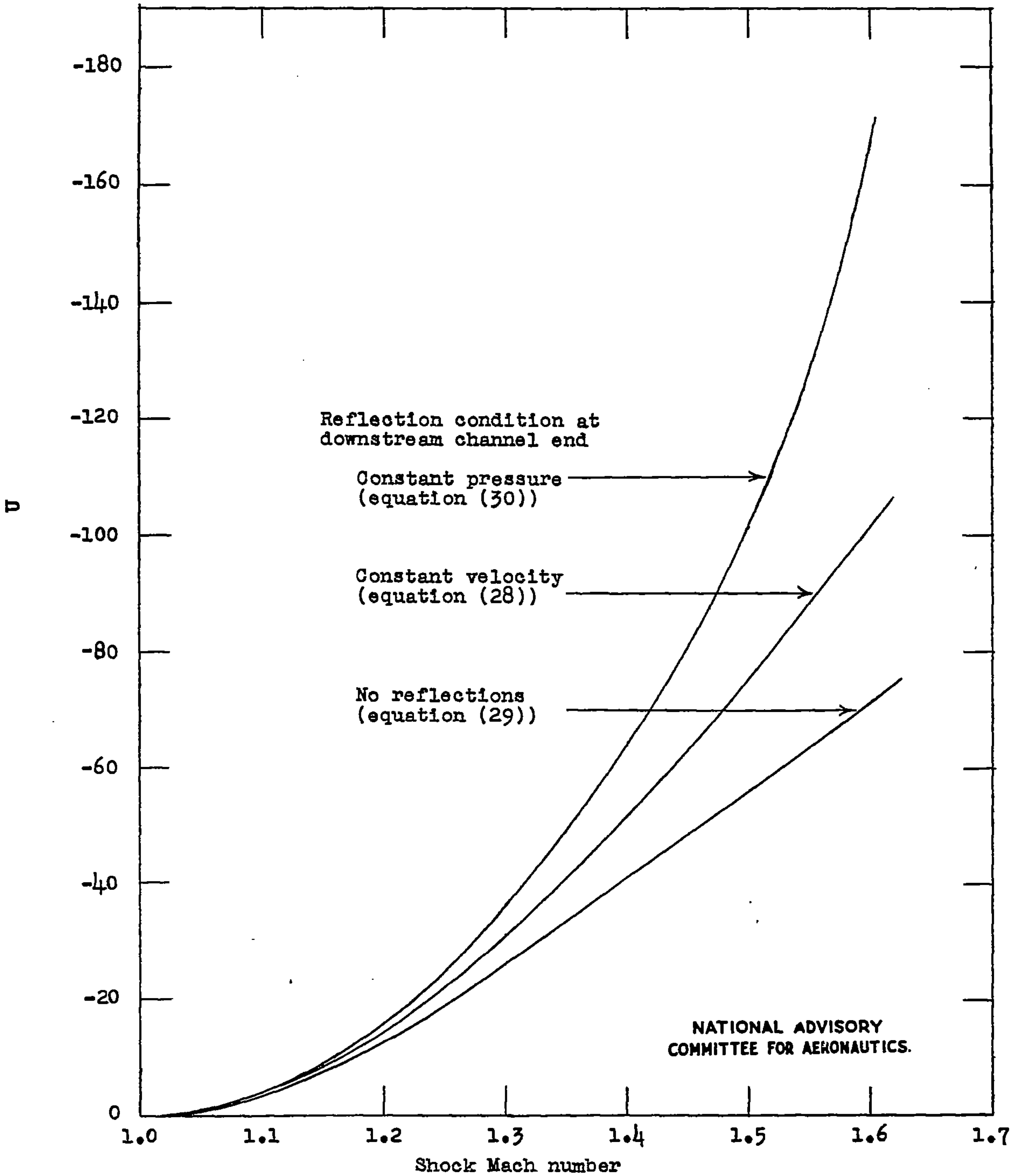

Figure 7.- The velocity of shock waves in trapped pulses for the case where the original equilibrium flow does not involve shock waves. The shock velocity is negative and thus, the shock always moves upstream, consuming expansion pulses and making compression pulses grow indefinitely. Computations are made for a_s = 1117 feet per second.

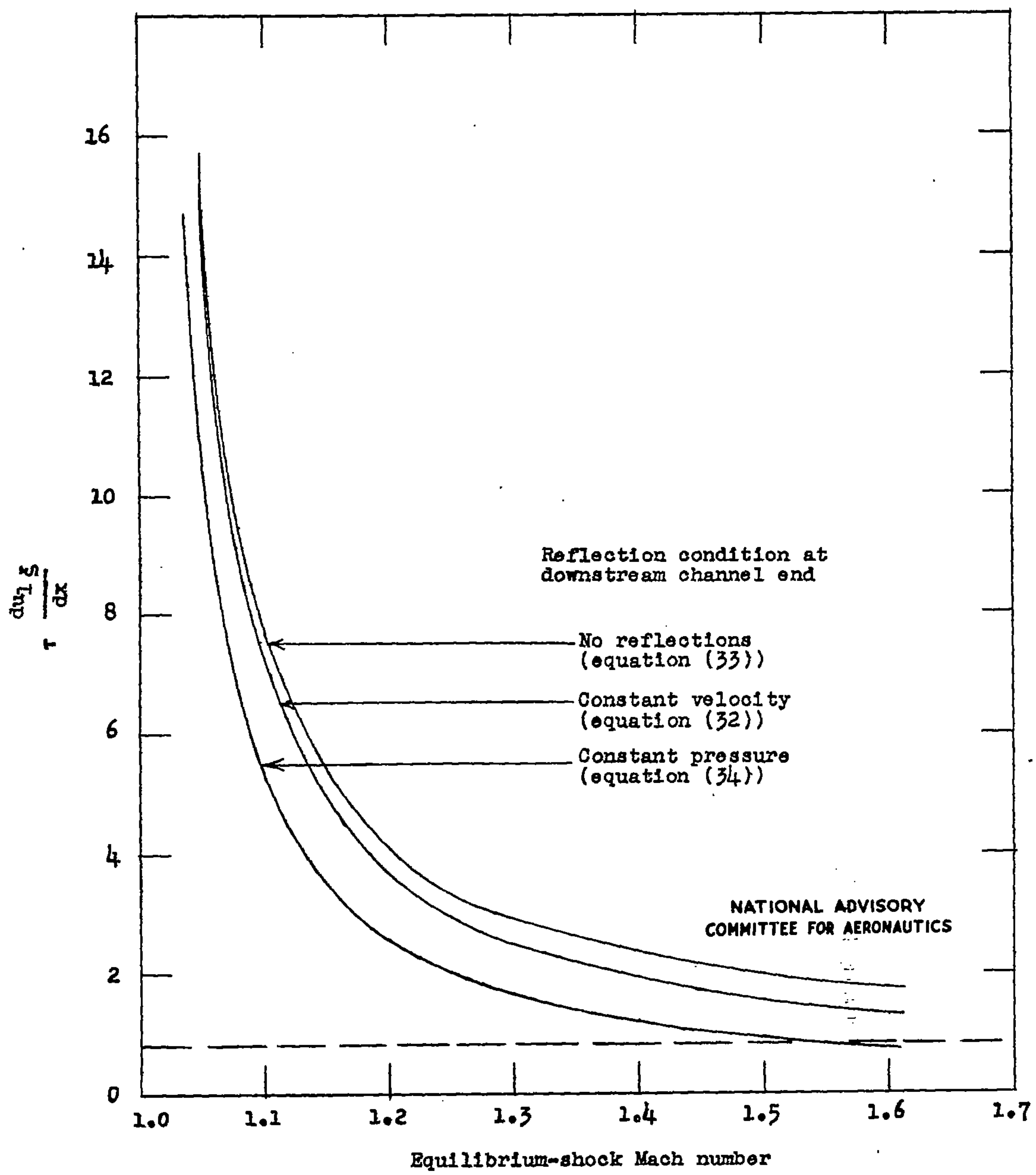

Figure 8.- The relaxation time for the return of a shock to its equilibrium position. The value of τ is computed from the linearized shock motion equation $\frac{d\xi}{dt} + \tau\xi = 0$. The value of τ is seen to be positive (stable shock position) for $\frac{du_{1\xi}}{dx} > 0$, that is, for diverging channels, and negative (unstable shock position) for converging channels. For comparison, the relaxation time for the approach of a pulse to the trapped state is $\frac{1}{-\frac{6}{5}\frac{du_o}{dx}}$ shown by the dashed line.

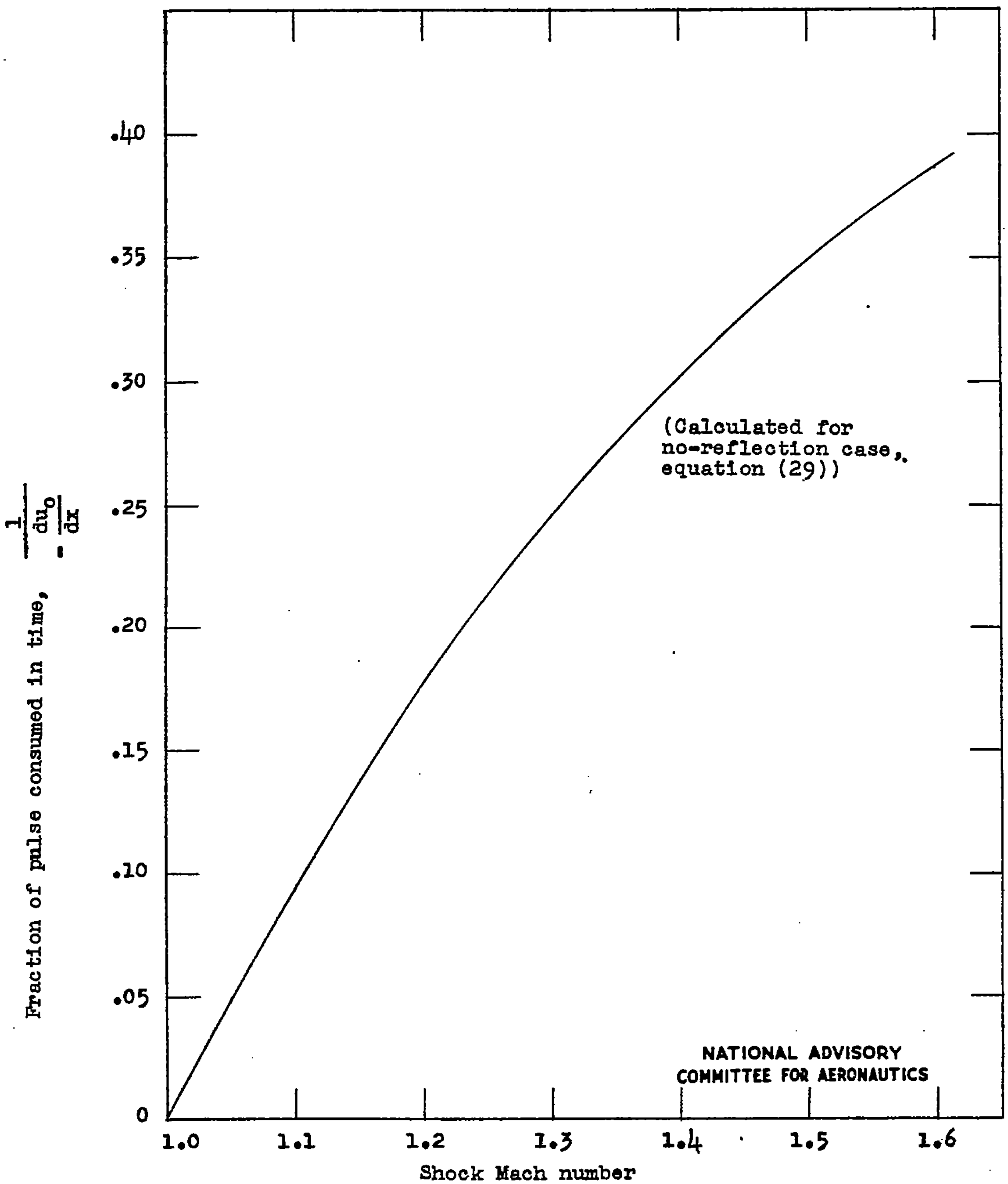

Figure 9.- Fraction of the area of a trapped pulse which is consumed in the time $\dfrac{1}{-\dfrac{du_o}{dx}}$. This result indicates that if the shock intensity is small enough, only a small fraction of the pulse is consumed before the pulse is fully trapped. In such cases, the division into trapping and consumption processes will be justified.

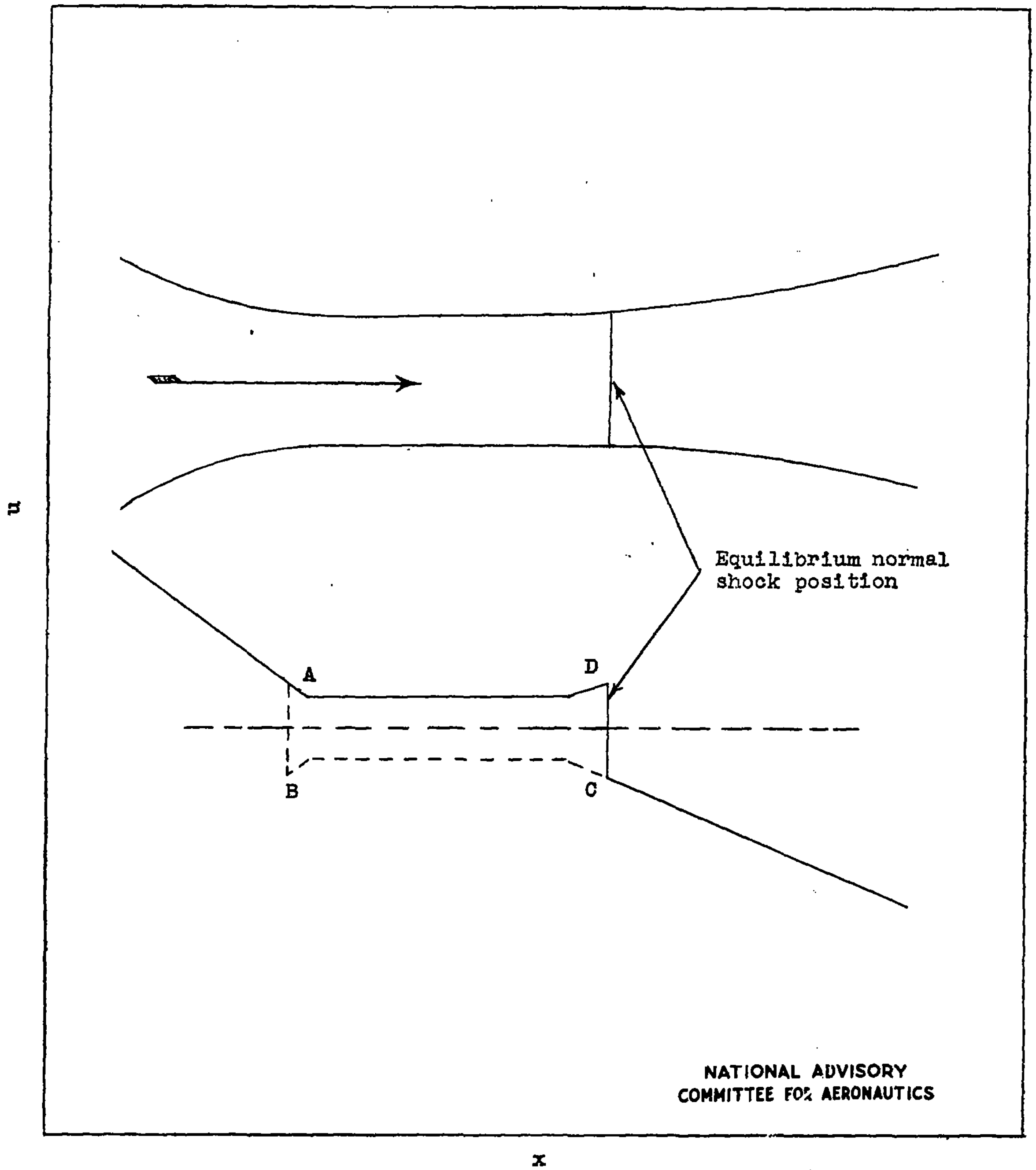

Figure 10.- Suggested long-throat supersonic diffuser. The limiting compression pulse that could be absorbed by this diffuser would have the pulse area, ABCD. Note that the pulse areas plotted here are $\frac{1}{2}\int Q\,dx$ since here $u' = \frac{Q}{2}$.

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